# Pseudo-random numbers: $\underbrace{\text{a}}_{\textit{mostly}}$ $\overbrace{\text{line}}^{\textit{of code}}$ at a time


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- Generation of unique cryptographic keys.
- Search and optimization via random walks.
- Selection: `quicksort` (C. A. R. Hoare, *ACM Algorithm 64: Quicksort*, Comm. ACM. **4**(7), 321, July 1961) was the first widely-used divide-and-conquer *algorithm* to reduce an $\mathcal{O}(N^2)$ problem to (on average) $\mathcal{O}(N \lg(N))$. Cf. Fast Fourier Transform (Gauss (1866) (Latin), Runge (1906), Danielson and Lanczos (crystallography) (1942), Cooley and Tukey (1965)).

# Historical note: al-Khwarizmi

Abu 'Abd Allah Muhammad ibn Musa al-Khwarizmi (ca. 780–850) is the father of *algorithm* and of *algebra*, from his book *Hisab Al-Jabr wal Mugabalah (Book of Calculations, Restoration and Reduction)*. He is celebrated in a 1200-year anniversary Soviet Union stamp:

# What are random numbers good for? ...

- Simulation.

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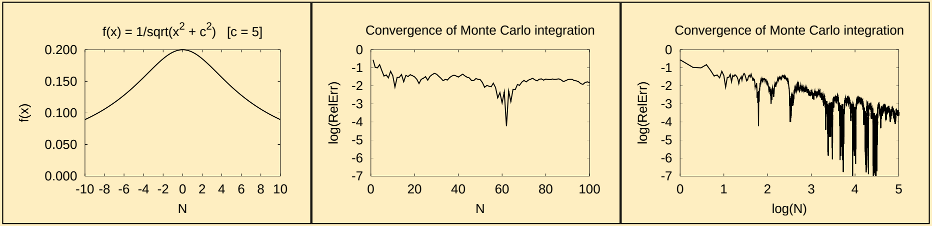
- Simulation.
- Sampling: unbiased selection of random data in statistical computations (opinion polls, experimental measurements, voting, Monte Carlo integration, ...). The latter is done like this ($x_k$ is random in $(a, b)$):

$$\int_a^b f(x)\,dx \approx \left( \frac{(b-a)}{N} \sum_{k=1}^{N} f(x_k) \right) + \mathcal{O}(1/\sqrt{N})$$

# Monte Carlo integration

Here is an example of a simple, smooth, and exactly integrable function, and the relative error of its Monte Carlo integration:



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- It isn't enough to conform to an expected distribution: the *order* that values appear in must be haphazard.
- Mathematical characterization of randomness is possible, but difficult.
- The best that we can usually do is *compute statistical measures of closeness* to particular expected distributions.

# Distributions of pseudo-random numbers

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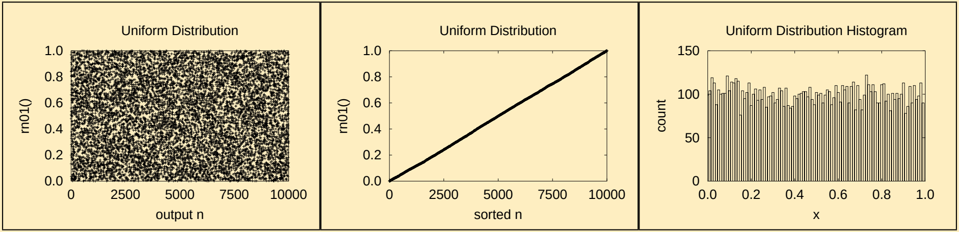
- Uniform (most common).
- Exponential.
- Normal (bell-shaped curve).
- Logarithmic: if $\mathrm{ran}()$ is uniformly-distributed in $(a, b)$, define $\mathrm{randl}(x) = \exp(x \,\mathrm{ran}())$. Then $a \,\mathrm{randl}(\ln(b/a))$ is logarithmically distributed in $(a, b)$. [Important use: sampling in floating-point number intervals.]

Sample logarithmic distribution:

```
% hoc
a = 1
b = 1000000
for (k = 1; k <= 10; ++k) printf "%16.8f\n", a*randl(ln(b/a))
    664.28612484
 199327.86997895
 562773.43156449
  91652.89169494
     34.18748767
    472.74816777
     12.34092778
      2.03900107
  44426.83813202
     28.79498121
```

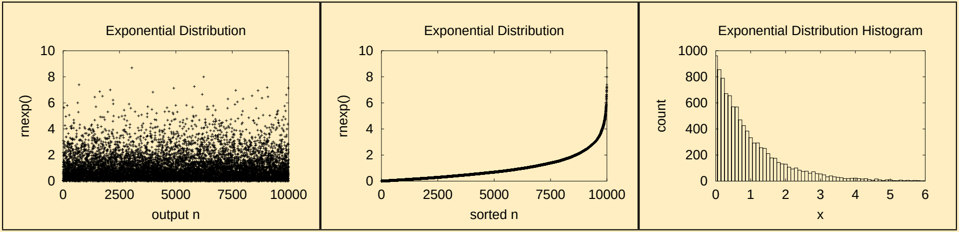
# Uniform distribution

Here are three ways to visualize a pseudo-random number distribution, using the Dyadkin-Hamilton generator function `rn01()`, which produces results uniformly distributed on $(0, 1]$:



# Exponential distribution

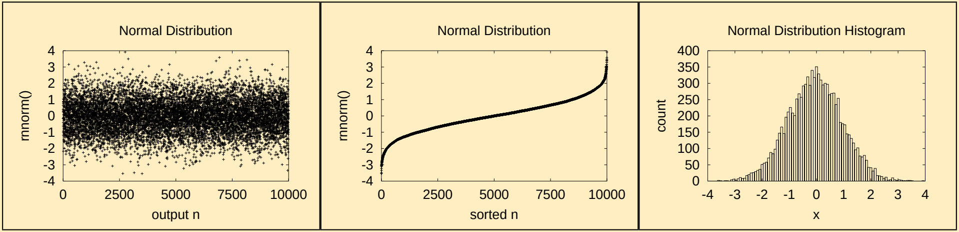
Here are visualizations of computations with the Dyadkin-Hamilton generator `rnexp()`, which produces results exponentially distributed on $[0, \infty)$:




Even though the theoretical range is $[0, \infty)$, the results are practically always modest: the probability of a result as big as 50 is smaller than $2 \times 10^{-22}$. At one result per microsecond, it could take 164 million years of computing to encounter such a value!

# Normal distribution

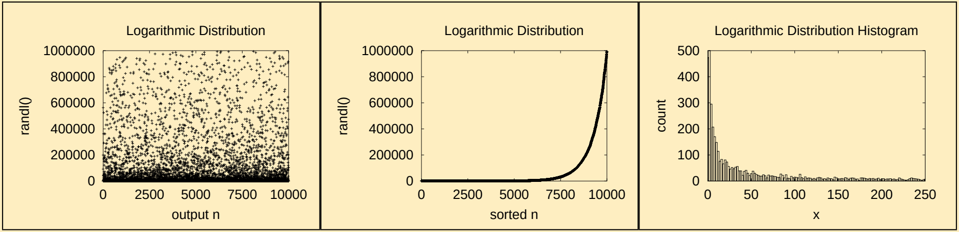
Here are visualizations of computations with the Dyadkin-Hamilton generator `rnnorm()`, which produces results normally distributed on $(-\infty, +\infty)$:




Results are never very large: a result as big as 7 occurs with probability smaller than $5 \times 10^{-23}$. At one result per microsecond, it could take 757 million years of computing to encounter such a value.

# Logarithmic distribution

Here are visualizations of computations with the `hoc` generator `randl(ln(1000000))`, which produces results normally distributed on $(1, 1000000)$:




The graphs are similar to those for the exponential distribution, but here, the result range is controlled by the argument of `randl()`.

# Goodness of fit: the $\chi^2$ measure

Given a set of $n$ *independent observations* with measured values $M_k$ and expected values $E_k$, then $\sum_{k=1}^{n} |(E_k - M_k)|$ is a measure of goodness of fit. So is $\sum_{k=1}^{n} (E_k - M_k)^2$. Statisticians use instead a measure introduced in 1900 by one of the founders of modern statistics, the English mathematician Karl Pearson (1857–1936):


(1880)

$$\chi^2 \text{ measure} = \sum_{k=1}^{n} \frac{(E_k - M_k)^2}{E_k}$$

Equivalently, if we have $s$ categories expected to occur with probability $p_k$, and if we take $n$ samples, counting the number $Y_k$ in category $k$, then

$$\chi^2 \text{ measure} = \sum_{k=1}^{s} \frac{(np_k - Y_k)^2}{np_k}$$

# Goodness of fit: the $\chi^2$ measure ...

The theoretical $\chi^2$ distribution depends on the number of degrees of freedom, and table entries look like this (highlighted entries are referred to later):

| D.o.f. | $p = 1\%$ | $p = 5\%$ | $p = 25\%$ | $p = 50\%$ | $p = 75\%$ | $p = 95\%$ | $p = 99\%$ |
|---|---|---|---|---|---|---|---|
| $\boldsymbol{\nu = 1}$ | 0.00016 | 0.00393 | 0.1015 | **0.4549** | 1.323 | 3.841 | **6.635** |
| $\nu = 5$ | 0.5543 | 1.1455 | 2.675 | 4.351 | 6.626 | 11.07 | 15.09 |
| $\boldsymbol{\nu = 10}$ | 2.558 | 3.940 | 6.737 | 9.342 | 12.55 | 18.31 | **23.21** |
| $\nu = 50$ | 29.71 | 34.76 | 42.94 | 49.33 | 56.33 | 67.50 | 76.15 |

For example, this table says:

> **For $\boldsymbol{\nu = 10}$, the probability that the $\chi^2$ measure is no larger than 23.21 is 99%.**
> **In other words, $\chi^2$ measures larger than 23.21 should occur only about 1% of the time.**

# Goodness of fit: coin-toss experiments

Coin toss has one degree of freedom, $\boldsymbol{\nu = 1}$, because if it is not heads, then it must be tails.

```
% hoc
for (k = 1; k <= 10; ++k) print randint(0,1), ""
0 1 1 1 0 0 0 0 1 0
```

This gave four 1s and six 0s:

$$
\begin{aligned}
\chi^2 \text{ measure} &= \frac{(10 \times 0.5 - 4)^2 + (10 \times 0.5 - 6)^2}{10 \times 0.5} \\
&= 2/5 \\
&= 0.40
\end{aligned}
$$

# Goodness of fit: coin-toss experiments ...

From the table, for $\boldsymbol{\nu = 1}$, we expect a $\chi^2$ measure no larger than **0.4549** half of the time, so our result is reasonable.
On the other hand, if we got nine 1s and one 0, then we have

$$\begin{aligned}\chi^2 \text{ measure} &= \frac{(10 \times 0.5 - 9)^2 + (10 \times 0.5 - 1)^2}{10 \times 0.5} \\ &= 32/5 \\ &= 6.4\end{aligned}$$

This is close to the tabulated value **6.635** at $p = 99\%$. That is, **we should only expect nine-of-a-kind about once in every 100 experiments.**
If we had all 1s or all 0s, the $\chi^2$ measure is 10 (probability $p = 0.998$) [**twice in 1000 experiments**].
If we had equal numbers of 1s and 0s, then the $\chi^2$ measure is 0, indicating an exact fit.

# Goodness of fit: coin-toss experiments ...

Let's try 100 similar experiments, counting the number of 1s in each experiment:

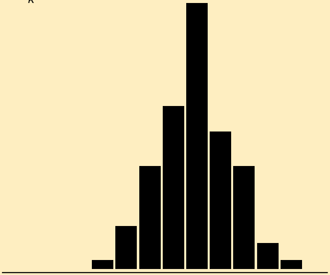
```
% hoc
for (n = 1; n <= 100; ++n) {
    sum = 0
    for (k = 1; k <= 10; ++k) \
        sum += randint(0,1)
    print sum, ""
}
4 4 7 3 5 5 5 2 5 6 6 6 3 6 6 7 4 5 4 5 5 4
3 6 6 9 5 3 4 5 4 4 4 5 4 5 5 4 6 3 5 5 3 4
4 7 2 6 5 3 6 5 6 7 6 2 5 3 5 5 5 7 8 7 3 7
8 4 2 7 7 3 3 5 4 7 3 6 2 4 5 1 4 5 5 5 6 6
5 6 5 5 4 8 7 7 5 5 4 5
```

# Goodness of fit: coin-toss experiments ...

The measured frequencies of the sums are:

**100 experiments**

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $Y_k$ | 0 | 1 | 5 | 12 | 19 | 31 | 16 | 12 | 3 | 1 | 0 |



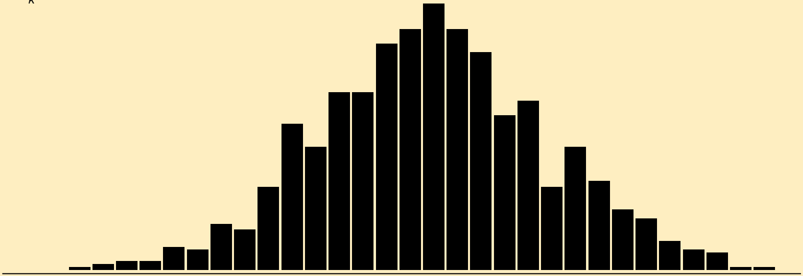
Notice that nine-of-a-kind occurred **once** each for 0s and 1s, as predicted.

# Goodness of fit: coin-toss experiments . . .

A simple one-character change on the outer loop limit produces the next experiment:

**1000 experiments**

| $k$ | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 | 65 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $Y_k$ | 1 | 2 | 3 | 3 | 8 | 7 | 16 | 14 | 29 | 51 | 43 | 62 | 62 | 79 | 84 | 93 | 84 | 76 | 54 | 59 | 29 | 43 | 31 | 21 | 18 | 10 | 7 | 6 | 1 | 1 | 0 |



# Goodness of fit: coin-toss experiments ...

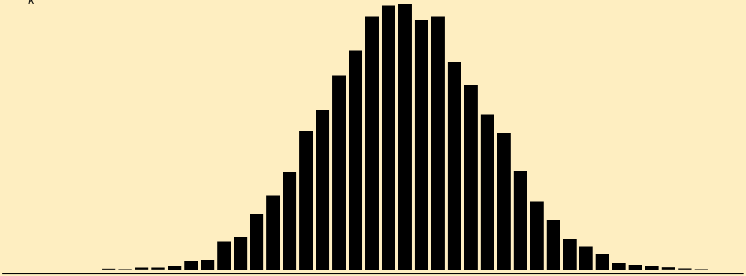
Another one-character change gives us this:

**10 000 experiments**

| $k$ | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $Y_k$ | 0 | 0 | 3 | 1 | 7 | 7 | 12 | 27 | 30 | 85 | 99 | 168 | 224 | 295 |

| $k$ | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $Y_k$ | 420 | 484 | 588 | 663 | 766 | 799 | 804 | 755 | 766 | 628 | 559 | 470 | 413 | 298 |

| $k$ | 58 | 59 | 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $Y_k$ | 207 | 150 | 93 | 70 | 48 | 20 | 15 | 12 | 8 | 4 | 1 | 0 | 0 |



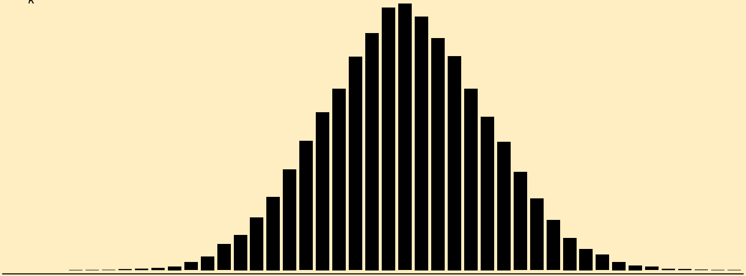
# Goodness of fit: coin-toss experiments ...

A final one-character change gives us this result for one million coin tosses:

**100 000 experiments**

| $k$ | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $Y_k$ | 1 | 4 | 12 | 34 | 47 | 70 | 118 | 242 | 418 | 802 | 1086 | 1633 | 2259 | 3112 | 3987 | 4880 | 5609 | 6587 | 7320 | 8113 | 8227 |

| $k$ | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $Y_k$ | 7828 | 7171 | 6607 | 5604 | 4740 | 3962 | 3029 | 2212 | 1544 | 996 | 654 | 474 | 257 | 141 | 107 | 43 | 37 | 21 | 5 | 5 |



# Are the digits of $\pi$ random?

Here are $\chi^2$ results for the digits of $\pi$ from recent computational records ( $\boldsymbol{\chi^2(\nu = 9, p = 0.99) \approx 21.67}$ ):

**$\pi$**

| Digits | Base | $\chi^2$ | $p(\chi^2)$ |
|---|---|---|---|
| 6B | 10 | 9.00 | 0.56 |
| 50B | 10 | 5.60 | 0.22 |
| 200B | 10 | 8.09 | 0.47 |
| 1T | 10 | 14.97 | 0.91 |
| 1T | 16 | 7.94 | 0.46 |

**$1/\pi$**

| Digits | Base | $\chi^2$ | $p(\chi^2)$ |
|---|---|---|---|
| 6B | 10 | 5.44 | 0.21 |
| 50B | 10 | 7.04 | 0.37 |
| 200B | 10 | 4.18 | 0.10 |

Whether the fractional digits of $\pi$, and most other transcendentals, are *normal* ($\approx$ equally likely to occur) is an outstanding unsolved problem in mathematics.

# The Central-Limit Theorem

The famous **Central-Limit Theorem** (de Moivre (1718), Laplace (1810), and Cauchy (1853)), says:

> **A suitably normalized sum of independent random variables is likely to be normally distributed, as the number of variables grows beyond all bounds. It is not necessary that the variables all have the same distribution function or even that they be wholly independent.**
>
> *— I. S. Sokolnikoff and R. M. Redheffer*
> *Mathematics of Physics and Modern Engineering, 2nd ed.*

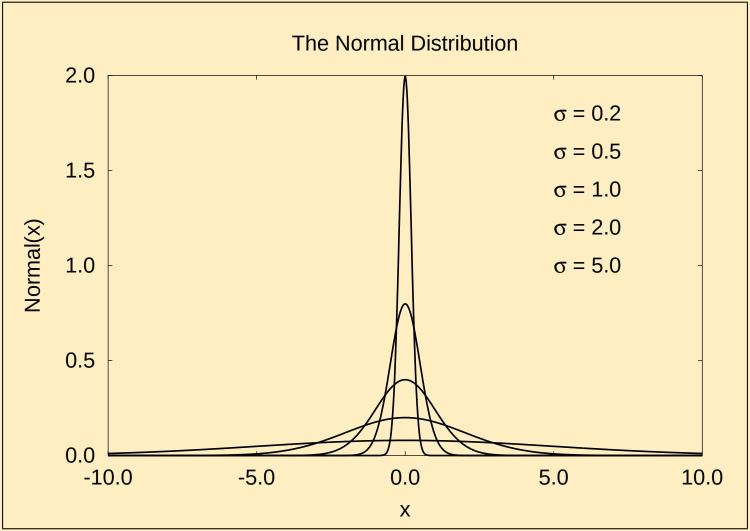
# The Central-Limit Theorem ...

In mathematical terms, this is

$$P(n\mu + r_1\sqrt{n} \leq X_1 + X_2 + \cdots + X_n \leq n\mu + r_2\sqrt{n})$$
$$\approx \frac{1}{\sigma\sqrt{2\pi}} \int_{r_1}^{r_2} \exp(-t^2/(2\sigma^2))\,dt$$

where the $X_k$ are independent, identically distributed, and bounded random variables, $\mu$ is their **mean value**, $\sigma$ is their **standard deviation**, and $\sigma^2$ is their **variance**.

The integrand of this probability function looks like this:



# The Central-Limit Theorem ...

The normal curve falls off very rapidly. We can compute its area in $[-x, +x]$ with a simple midpoint quadrature rule like this:

```
func f(x) {
    global sigma;
    return (1/(sigma*sqrt(2*PI)))* exp(-x*x/(2*sigma**2))
}

func q(a,b) {
    n = 10240
    h = (b - a)/n
    area = 0
    for (k = 0; k < n; ++k) \
        area += h*f(a + (k + 0.5)*h);
    return area
}
```

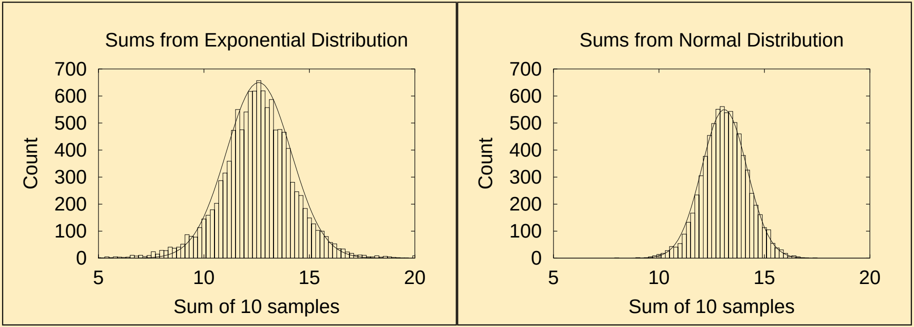
# The Central-Limit Theorem ...

```
sigma = 3
for (k = 1; k < 8; ++k) \
    printf "%d  %.9f\n", k, q(-k*sigma,k*sigma)
1  0.682689493
2  0.954499737
3  0.997300204
4  0.999936658
5  0.999999427
6  0.999999998
7  1.000000000
```

In computer management, **99.999% (five 9's) availability** is **five minutes downtime per year**.
In manufacturing, Motorola's **6$\sigma$ reliability** with 1.5$\sigma$ drift is about **three defects per million** (from $q(-(6-1.5)*\sigma, +(6-1.5)*\sigma)/2$).

# The Central-Limit Theorem . . .

It is remarkable that the Central-Limit Theorem applies also to nonuniform distributions. Here is a demonstration with sums from exponential and normal distributions:




Superimposed on the histograms are rough fits by eye of normal distribution curves $650\,\exp(-(x-12.6)^2/4.7)$ and $550\,\exp(-(x-13.1)^2/2.3)$.

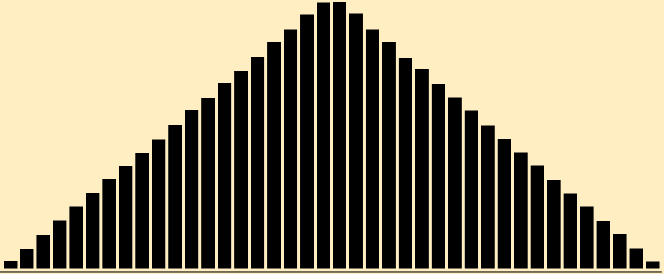
# The Central-Limit Theorem . . .

Not everything looks like a normal distribution. Here is a similar experiment, using *differences* of successive pseudo-random numbers, bucketizing them into 40 bins from the range $[-1.0, +1.0]$:

**10 000 experiments (counts scaled by 1/100)**

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $Y_k$ | 13 | 35 | 61 | 88 | 113 | 138 | 163 | 187 | 211 | 236 | 262 | 290 | 312 | 339 | 361 | 387 | 414 | 437 | 464 | 487 |

| $k$ | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $Y_k$ | 487 | 467 | 437 | 414 | 385 | 365 | 337 | 312 | 288 | 261 | 236 | 212 | 188 | 162 | 137 | 113 | 87 | 63 | 36 | 12 |



This one is known from theory: it is a *triangular* distribution. A similar result is obtained if one takes pair sums instead of differences.

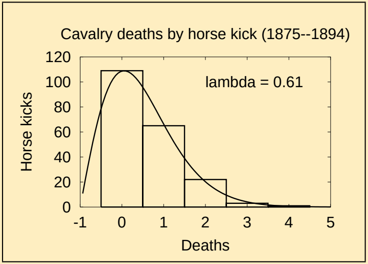
# Digression: Poisson distribution

The *Poisson* distribution arises in time series when the probability of an event occurring in an arbitrary interval is proportional to the length of the interval, and independent of other events:

$$P(X = n) = \frac{\lambda^n}{n!} e^{-\lambda}$$

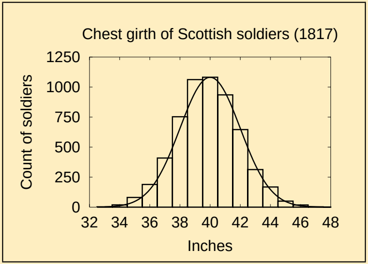
In 1898, Ladislaus von Bortkiewicz collected Prussian army data on the number of soldiers killed by horse kicks in 10 cavalry units over 20 years: 122 deaths, or an average of $122/200 = 0.61$ deaths per unit per year.

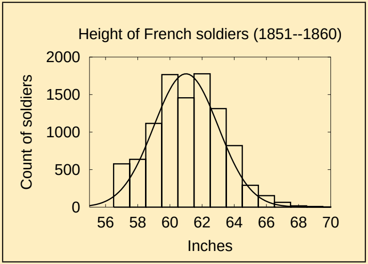
| Deaths | Kicks (actual) | $\lambda = 0.61$ Kicks (Poisson) |
|---|---|---|
| 0 | 109 | 108.7 |
| 1 | 65 | 66.3 |
| 2 | 22 | 20.2 |
| 3 | 3 | 4.1 |
| 4 | 1 | 0.6 |

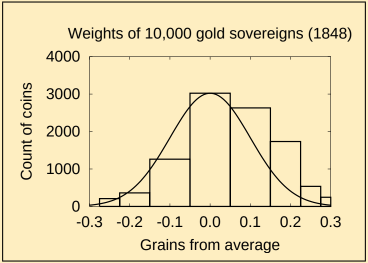


# The Central-Limit Theorem . . .

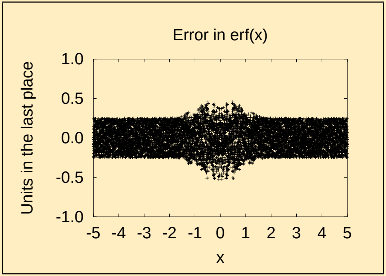
Measurements of physical phenomena often form normal distributions:

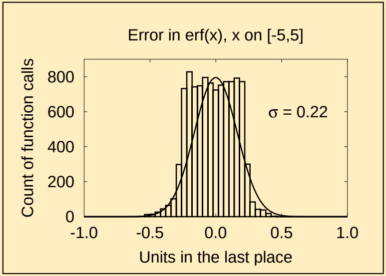


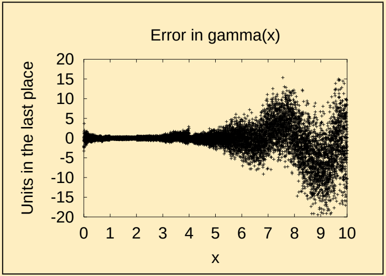



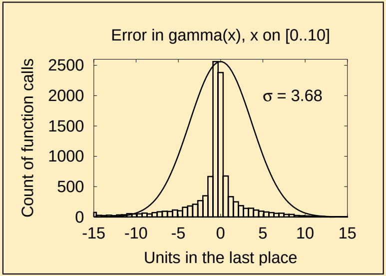



# The Central-Limit Theorem ...



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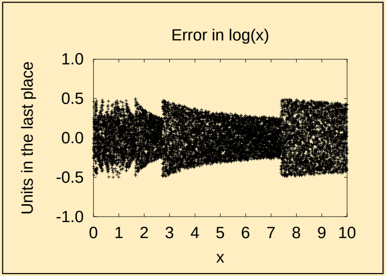
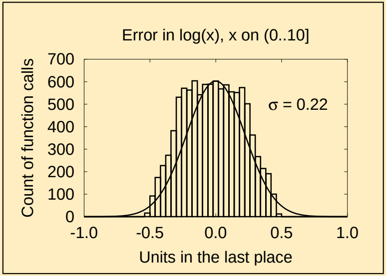
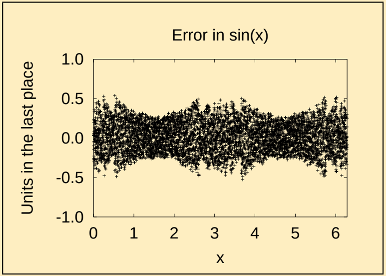
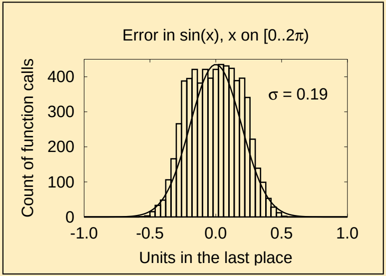


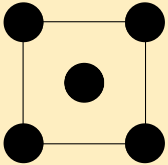



# The Normal Curve and Carl-Friedrich Gauß (1777–1855)

# The Normal Curve and the Quincunx

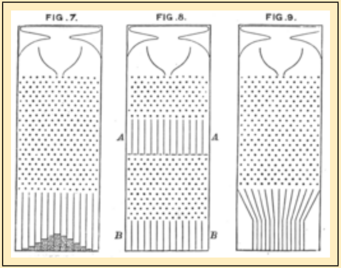


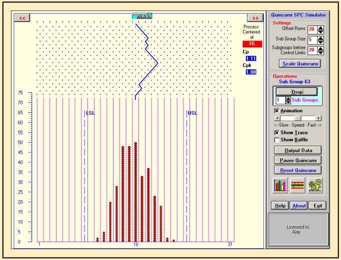
**quincunx**, *n.*

**2**. An arrangement or disposition of five objects so placed that four occupy the corners, and the fifth the centre, of a square or other rectangle; a set of five things arranged in this manner.
**b**. spec. as a basis of arrangement in planting trees, either in a single set of five or in combinations of this; a group of five trees so planted.

Oxford English Dictionary

# The Normal Curve and the Quincunx . . .




For simulations and other material on the quincunx (Galton's *bean machine*), see:

- `http://www.ms.uky.edu/~mai/java/stat/GaltonMachine.html`
- `http://www.rand.org/statistics/applets/clt.html`
- `http://www.stattucino.com/berrie/dsl/Galton.html`
- `http://teacherlink.org/content/math/interactive/flash/quincunx/quincunx.html`
- `http://www.bun.kyoto-u.ac.jp/~suchii/quinc.html`

# Remarks on random numbers

**Any one who considers arithmetical methods of producing random numbers is, of course, in a state of sin.**
— John von Neumann (1951)
[*The Art of Computer Programming, Vol. 2, Seminumerical Algorithms*, 3rd ed., p. 1]

**He talks at random; sure, the man is mad.**
— Queen Margaret
[William Shakespeare's *1 King Henry VI*, Act V, Scene 3 (1591)]

**A random number generator chosen at random isn't very random.**
— Donald E. Knuth (1997)
[*The Art of Computer Programming, Vol. 2, Seminumerical Algorithms*, 3rd ed., p. 384]

# How do we generate pseudo-random numbers?

- Linear-congruential generators (most common):
  $r_{n+1} = (ar_n + c) \bmod m$, for integers $a$, $c$, and $m$, where $0 < m$, $0 \le a < m$, $0 \le c < m$, with starting value $0 \le r_0 < m$.

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- Multiplicative (bad):
  $r_{n+1} = (r_{n-\alpha} \times r_{n-\beta}) \bmod m$.
- Shift register:
  $r_{n+k} = \sum_{i=0}^{k-1}(a_i r_{n+i} \pmod 2) \qquad (a_i = 0, 1)$.

# How do we generate pseudo-random numbers? ...

Given an integer $r \in [A, B)$, $x = (r - A)/(B - A + 1)$ is on $[0, 1)$.
However, interval reduction by $A + (r - A) \bmod s$ to get a distribution in $(A, C)$, where $s = (C - A + 1)$, is possible only for certain values of $s$.
Consider reduction of $[0, 4095]$ to $[0, m]$, with $m \in [1, 9]$: we get equal distribution of remainders only for $m = 2^q - 1$:

| | $m$ | counts of remainders $k$ mod $(m+1)$, $k \in [0, m]$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| OK | 1 | 2048 | 2048 | | | | | | | | |
| | 2 | **1366** | 1365 | 1365 | | | | | | | |
| OK | 3 | 1024 | 1024 | 1024 | 1024 | | | | | | |
| | 4 | **820** | 819 | 819 | 819 | 819 | | | | | |
| | 5 | **683** | **683** | **683** | **683** | 682 | 682 | | | | |
| | 6 | **586** | 585 | 585 | 585 | 585 | 585 | 585 | | | |
| OK | 7 | 512 | 512 | 512 | 512 | 512 | 512 | 512 | 512 | | |
| | 8 | **456** | 455 | 455 | 455 | 455 | 455 | 455 | 455 | 455 | |
| | 9 | **410** | **410** | **410** | **410** | **410** | **410** | 409 | 409 | 409 | 409 |

# How do we generate pseudo-random numbers? ...

Samples from other distributions can usually be obtained by some suitable transformation. Here is the simplest generator for the normal distribution, assuming that `randu()` returns uniformly-distributed values on $(0, 1]$:

```
func randpmnd() \
{ ## Polar method for random deviates
  ## Algorithm P, p. 122, from Donald E. Knuth,
  ## The Art of Computer Programming, vol. 2, 3/e, 1998
  while (1) \
  {
    v1 = 2*randu() - 1  # v1 on [-1,+1]
    v2 = 2*randu() - 1  # v2 on [-1,+1]
    s = v1*v1 + v2*v2   # s on [0,2]
    if (s < 1) break    # exit loop if s inside unit circle
  }
  return (v1 * sqrt(-2*ln(s)/s))
}
```

# Period of a sequence

All pseudo-random number generators eventually reproduce the starting sequence; the *period* is the number of values generated before this happens.

Widely-used historical generators have periods of a few tens of thousands to a few billion, but good generators are now known with very large periods:

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| $> 10^{14100}$ | Deng and Xu (2003). |

# Reproducible sequences

In computational applications with pseudo-random numbers, it is *essential* to be able to reproduce a previous calculation. Thus, generators are required that can be set to a given **initial seed**:

```
% hoc
for (k = 0; k < 3; ++k) \
{
    setrand(12345)
    for (n = 0; n < 10; ++n) print int(rand()*100000),""
    println ""
}
88185 5927 13313 23165 64063 90785 24066 37277 55587 62319
88185 5927 13313 23165 64063 90785 24066 37277 55587 62319
88185 5927 13313 23165 64063 90785 24066 37277 55587 62319
```

# Reproducible sequences . . .

If the seed is not reset, different sequences are obtained for each test run. Here is the same code as before, with the `setrand()` call disabled:

```
for (k = 0; k < 3; ++k) \
{
    ## setrand(12345)
    for (n = 0; n < 10; ++n) print int(rand()*100000),""
    println ""
}
36751 37971 98416 59977 49189 85225 43973 93578 61366 54404
70725 83952 53720 77094 2835 5058 39102 73613 5408 190
83957 30833 75531 85236 26699 79005 65317 90466 43540 14295
```

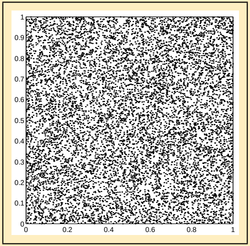
In practice, **software must have its own source-code implementation of the generators**: vendor-provided ones do *not* suffice.

# The correlation problem

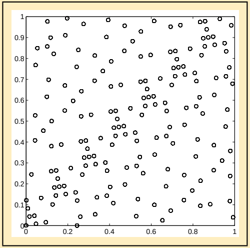
**Random numbers fall mainly in the planes**
— *George Marsaglia (1968)*

Linear-congruential generators are known to have correlation of successive numbers: if these are used as coordinates in a graph, one gets patterns, instead of uniform grey:

**Good**




**Bad**




**The number of points plotted is the same in each graph.**

# The correlation problem . . .

The good generator is Matlab's `rand()`. Here is the bad generator:

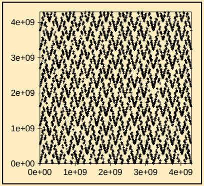
```
% hoc
func badran() {
    global A, C, M, r;
    r = int(A*r + C) % M;
    return r }
M = 2^15 - 1; A = 2^7 - 1 ; C = 2^5 - 1
r = 0 ; r0 = r ; s = -1 ; period = 0

while (s != r0) {period++; s = badran(); print s, "" }
    31 3968 12462 9889 10788 26660 ... 22258 8835 7998 0

# Show the sequence period
println period
    175

# Show that the sequence repeats
for (k = 1; k <= 5; ++k) print badran(),""
    31 3968 12462 9889 10788
```

# The correlation problem . . .

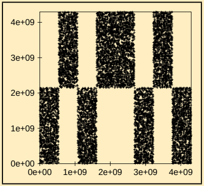
Marsaglia's (2003) family of xor-shift generators:

```
y ^= y << a; y ^= y >> b; y ^= y << c;
```

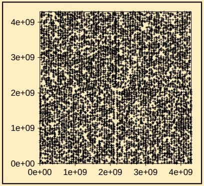
I-003



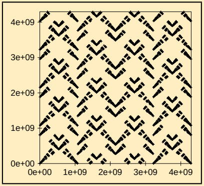

I-007




I-028




I-077



# Generating random integers

When the endpoints of a floating-point uniform pseudo-random number generator are uncertain, generate random integers in [low,high] like this:

```
func irand(low, high) \
{
    # Ensure integer endpoints
    low = int(low)
    high = int(high)

    # Sanity check on argument order
    if (low >= high) return (low)

    # Find a value in the required range
    n = low - 1
    while ((n < low) || (high < n)) \
        n = low + int(rand() * (high + 1 - low))

    return (n)
}


for (k = 1; k <= 20; ++k) print irand(-9,9), ""
-9 -2 -2 -7 7 9 -3 0 4 8 -3 -9 4 7 -7 8 -3 -4 8 -4

for (k = 1; k <= 20; ++k) print irand(0, 10^6), ""
986598 580968 627992 379949 700143 734615 361237
322631 116247 369376 509615 734421 321400 876989
940425 139472 255449 394759 113286 95688
```

# Generating random integers in order

```
% hoc
func bigrand() { return int(2^31 * rand()) }

# select(m,n): select m pseudo-random integers from (0,n) in order
proc select(m,n) \
{
    mleft = m
    remaining = n
    for (i = 0; i < n; ++i) \
    {
        if (int(bigrand() % remaining) < mleft) \
        {
            print i, ""
            mleft--
        }
        remaining--
    }
    println ""
}
```

See Chapter 12 of Jon Bentley, *Programming Pearls*, 2nd ed., Addison-Wesley (2000), ISBN 0-201-65788-0. [ACM TOMS **6**(3), 359–364, September 1980].

# Generating random integers in order . . .

Here is how the `select()` function works:

```
select(3,10)
5 6 7

select(3,10)
0 7 8

select(3,10)
2 5 6

select(3,10)
1 5 7

select(10,100000)
7355 20672 23457 29273 33145 37562 72316 84442 88329 97929

select(10,100000)
401 8336 41917 43487 44793 56923 61443 90474 92112 92799
```

# Testing pseudo-random number generators

Most tests are based on computing a $\chi^2$ measure of computed and theoretical values.

**If one gets values p $< 1\%$ or p $> 99\%$ for several tests, the generator is suspect.**

Marsaglia Diehard Battery test suite (1985): 15 tests.

Marsaglia/Tsang `tuftest` suite (2002): 3 tests.

All produce *p* values that can be checked for reasonableness.

These tests all expect *uniformly-distributed* pseudo-random numbers.

# Testing nonuniform pseudo-random number generators

How do you test a generator that produces pseudo-random numbers in some other distribution? You have to figure out a way to use those values to produce an expected uniform distribution that can be fed into the standard test programs.

For example, take the negative log of exponentially-distributed values, since $-\log(\exp(-\text{random})) = \text{random}$.

For normal distributions, consider successive pairs $(x, y)$ as a 2-dimensional vector, and express in polar form $(r, \theta)$: $\theta$ is then uniformly distributed in $[0, 2\pi)$, and $\theta/(2\pi)$ is in $[0, 1)$.

# The Marsaglia/Tsang `tuftest` tests

Just three tests instead of the fifteen of the Diehard suite:

- b'day test (generalization of Birthday Paradox).

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- ❏ Euclid's (ca. 330–225BC) gcd test.

# The Marsaglia/Tsang `tuftest` tests

Just three tests instead of the fifteen of the Diehard suite:

- b'day test (generalization of Birthday Paradox).
- Euclid's (ca. 330–225BC) gcd test.
- Gorilla test (generalization of monkey's typing random streams of characters).

# Digression: The Birthday Paradox

The *birthday paradox* arises from the question **How many people do you need in a room before the probability is at least half that two of them share a birthday?**

The answer is just 23, not $365/2 = 182.5$.

The probability that *none* of $n$ people are born on the same day is

$$\begin{aligned} P(1) &= 1 \\ P(n) &= P(n-1) \times (365 - (n-1))/365 \end{aligned}$$

The $n$-th person has a choice of $365 - (n-1)$ days to not share a birthday with any of the previous ones. Thus, $(365 - (n-1))/365$ is the probability that the $n$-th person is not born on the same day as any of the previous ones, assuming that they are born on different days.

# Digression: The Birthday Paradox ...

Here are the probabilities that $n$ people *share* a birthday (i.e., $1 - P(n)$):

```
% hoc128
PREC = 3
p = 1
for (n = 1;n <= 365;++n) \
    {p *= (365-(n-1))/365; println n,1-p}
1 0
2 0.00274
3 0.00820
4 0.0164
...
22 0.476
23 0.507
24 0.538
...
100 0.999999693
...
```

$P(365) \approx 1.45 \times 10^{-157}$ [cf. $10^{80}$ particles in universe].

# Digression: Euclid's algorithm (ca. 300BC)

This is the oldest surviving nontrivial algorithm in mathematics.

```
func gcd(x,y) \
{ ## greatest common denominator of integer x, y
  r = abs(x) % abs(y)
  if (r == 0) return abs(y) else return gcd(y, r)
}

func lcm(x,y) \
{ ## least common multiple of integer x,y
  x = int(x)
  y = int(y)
  if ((x == 0) || (y == 0)) return (0)
  return ((x * y)/gcd(x,y))
}
```

# Digression: Euclid's algorithm ...

Complete rigorous analysis of Euclid's algorithm was not achieved until 1970–1990!

The average number of steps is

$$\begin{aligned} A\left(\gcd(x, y)\right) &\approx \left((12 \ln 2) / \pi^2\right) \ln y \\ &\approx 1.9405 \log_{10} y \end{aligned}$$

and the maximum number is

$$\begin{aligned} M\left(\gcd(x, y)\right) &= \left\lfloor \log_{\phi}\left((3 - \phi) y\right) \right\rfloor \\ &\approx 4.785 \log_{10} y + 0.6723 \end{aligned}$$

where $\phi = (1 + \sqrt{5})/2 \approx 1.6180$ is the golden ratio.