A Note for Problem No. 2 (Updated)

In the following, we make a suggestion in several steps for implementing the MCMC approach for the project. The goal is to generate samples from a distribution in the form of histogram of the VIX daily returns that is estimated from previous returns.

1. Download the data set from the past year, form a collection of daily returns and plot the corresponding histogram. This histogram, we should denote by the vector $\pi(j), j = 1, \ldots, M$, serves as the target distribution we would like to simulate. Here we assume there are M possible states for the returns, each covering an interval for the values of the returns. For the purpose of learning MCMC, a Markov chain with around 10 states is a reasonable choice. You may also clean up the data by removing several obvious outliers.

2. For each simulated Markov chain, we start with the current return, which resides in one of the states i, the new state is chosen according to the Metropolis-Hastings algorithm:

   (a) First we need to choose a proposal density $q(x_t, x_{t+1})$, which gives the trial transition probability from the state at t, x_t, to the state at $t+1$, x_{t+1}. With M states, it will be a $M \times M$ matrix. The entries $q_{i,j}$ are calculated based on your proposed jump scheme. For example, we propose the following scheme for moving from one state to another:

      i. Divide the return interval, such as $[-0.3, 0.3]$, into M subintervals each with length a;

      ii. For any return falling into subinterval i, we say that it is in state i;

      iii. Sample a normally distributed random number Y with mean 0 and variance σ^2;

      iv. Move from the center of i-th state by Y (to the right if $Y > 0$ and to the left if $Y < 0$), and check which interval it falls into;

      v. If it moves beyond the rightest interval (state M), keep it in that interval (state M); if it moves beyond the leftist interval (state 1), keep it in state 1;

      vi. With the above rules, calculate the probability of starting in state i and ending in state j, this is your $q_{i,j}$.

   (b) Now we can compute

   $$\alpha = \min\left(\frac{\pi(j)q_{i,j}}{\pi(i)q_{j,i}}, 1\right)$$

   where state j is the proposed new state according to the scheme suggested above.

(c) we accept the new state j with probability α, otherwise let it stay in state i

(d) repeat this procedure for N steps, record the end states after N steps.

(e) With the collection of end states for all the Markov chains simulated, we can construct a histogram and compare with π.