# Solution Notes for Homework Assignment 6

1. (a)

$$\begin{aligned}
d\left(e^{\beta t}R(t)\right) &= \beta e^{\beta t}R(t)dt + e^{\beta t}dR(t) \\
&= \beta e^{\beta t}R(t)dt + e^{\beta t}\left[\left(\alpha - \beta R(t)\right)dt + \sigma\, dW(t)\right] \\
&= e^{\beta t}\left[\alpha\, dt + \sigma\, dW(t)\right]
\end{aligned}$$

(b) Integrating from 0 to $t$,

$$\begin{aligned}
e^{\beta t}R(t) - R(0) &= \int_0^t e^{\beta s}\left[\alpha\, ds + \sigma\, dW(s)\right] \\
&= \alpha \int_0^t e^{\beta s}\, ds + \sigma \int_0^t e^{\beta s}\, dW(s) \\
&= \frac{\alpha}{\beta}\left(e^{\beta t} - 1\right) + \sigma \int_0^t e^{\beta s}\, dW(s).
\end{aligned}$$

Therefore

$$R(t) = e^{-\beta t}R(0) + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + \sigma \int_0^t e^{-\beta(t-s)}\, dW(s)$$

2. Obviously $B_1$ is a Brownian motion. To show that for $B_2$, we note

$$dB_2(t) = \rho(t)\, dW_1(t) + \sqrt{1-\rho^2(t)}\, dW_2(t)$$

which can be viewed as a sum of two independent normal random variables, so the sum is also normal, with mean zero, and variance

$$\rho(t)^2 dt + \left(\sqrt{1-\rho^2(t)}\right)^2 dt = dt$$

We also need to check that $B_2(t_2) - B_2(t_1)$ and $B_2(s_2) - B_2(s_1)$ are independent for $t_2 > t_1 > s_2 > s_1$. Finally we calculate

$$\begin{aligned}
dB_1(t)dB_2(t) &= dW_1(t) \cdot \left(\rho(t)\, dW_1(t) + \sqrt{1-\rho^2(t)}\, dW_2(t)\right) \\
&= \rho(t)\left(dW_1\right)^2 + \sqrt{1-\rho^2(t)}\, dW_1 \cdot dW_2 \\
&= \rho(t)dt
\end{aligned}$$

3. (a) Apply Itô's formula to $f(x,t) = e^{-\theta x - (r + \frac{1}{2}\theta^2)t}$,

$$\begin{aligned} d\zeta(t) &= f_t\, dt + f_x\, dW(t) + \frac{1}{2} f_{xx}\, dt \\ &= -\zeta(t) \left( r + \frac{1}{2}\theta^2 \right) dt + \zeta(t)(-\theta)\, dW(t) + \frac{1}{2}\zeta(t)(-\theta)^2 dt \\ &= -\theta\zeta(t)\, dW(t) - r\zeta(t)\, dt. \end{aligned}$$

(b) From the definition of $X(t)$, we have

$$\begin{aligned} dX(t) &= \Delta(t) dS(t) + r\left(X(t) - \Delta(t)S(t)\right) dt \\ &= \left[\alpha\Delta(t)S(t) + r\left(X(t) - \Delta(t)S(t)\right)\right] dt + \sigma\Delta(t)S(t)dW(t) \\ &= \sigma\left[\theta\Delta(t)S(t) + \frac{r}{\sigma}X(t)\right] dt + \sigma\Delta(t)S(t)dW(t) \end{aligned}$$

Using Itô's product rule,

$$d(\zeta(t)X(t)) = \zeta(t)dX(t) + X(t)d\zeta(t) + d\zeta(t)\cdot dX(t).$$

We collect the coefficients for $dt$ terms:

$$\zeta(t)\sigma\left[\theta\Delta(t)S(t) + \frac{r}{\sigma}X(t)\right] - rX(t)\zeta(t) - \sigma\theta\zeta(t)\Delta(t)S(t) = 0.$$

So $d(\zeta(t)X(t))$ only has $dW(t)$ terms, which means that $\zeta X$ is a martingale.

(c) We have demonstrated that by choosing $\Delta(t)$ properly, we can replicate the payoff of a derivative $V(T) = F(S(T))$ with a portfolio $X(t)$ such that

$$X(T) = V(T),$$

and based on no-arbitrage principle, we must have $X(t) = V(t)$ for all $0 \le t \le T$. In particular, if $\zeta(t)X(t)$ is a martingale, we have for $t = 0$

$$X(0) = \zeta(0)X(0) = E\left[\zeta(T)X(T)\right] = E\left[\zeta(T)V(T)\right].$$

This formula expresses the initial value of the portfolio, therefore the derivative price at $t = 0$, as an expected value of a random variable under the original probability measure. Note that this differs from the risk-neutral pricing formula in that not only the discount factor is included in $\zeta(t)$, but also a factor that resembles the R-N derivative we saw before.