Homework Assignment 6, Math 5765, due March 29

1. The Vasicek interest rate stochastic differential equation for the interest rate $R(t)$ is
$$dR(t) = (\alpha - \beta R(t))\, dt + \sigma\, dW(t),$$
where α, β and σ are positive constants. The solution to this equation can be derived in the following steps.

   (a) Use the product rule to compute $d(e^{\beta t}R(t))$. Notice that the right-hand-side now does not involve the unknown $R(t)$.

   (b) Integrate the equation in the above step to solve for $R(t)$.

2. Suppose $W_1(t)$ and $W_2(t)$ are independent Brownian motions and we introduce
$$\begin{aligned} B_1(t) &= W_1(t) \\ B_2(t) &= \int_0^t \rho(s)\, dW_1(s) + \int_0^t \sqrt{1-\rho^2(s)}\, dW_2(s) \end{aligned}$$
where $-1 \le \rho \le 1$ is a given function. Show that B_1 and B_2 are also Brownian motions and they satisfy
$$dB_1(t)\, dB_2(t) = \rho(t)\, dt$$

3. Let a stock price be a geometric Brownian motion
$$\frac{dS(t)}{S(t)} = \alpha\, dt + \sigma\, dW(t),$$
and let r denote the interest rate. The so-called market price of risk is
$$\theta = \frac{\alpha - r}{\sigma}$$
and the state price density process is
$$\zeta(t) = \exp\left\{-\theta W(t) - (r + \frac{1}{2}\theta^2)t\right\}.$$

   (a) Show that
$$d\zeta(t) = -\theta\zeta(t)\, dW(t) - r\zeta(t)\, dt.$$

   (b) Let X denote the value of a portfolio consisting of $\Delta(t)$ shares of the stock and a money market account. Show that $\zeta(t)X(t)$ is a martingale (under the original probability measure P).

(c) If the investor wants to start with $X(0)$ in a portfolio and end with $V(T) = F(S(T))$ where F is a given payoff function, show that he/she must begin with

$$X(0) = E\left[\zeta(T)V(T)\right].$$

Note that this expectation is taken in the original probability measure, which justifies calling $\zeta(t)$ the state price density process.