Solutions to Homework Assignment 5

1. (a) As the discounted stock price $e^{-rt}S(t)$ is a martingale under the risk-neutral measure,

$$e^{-rt}S(t) = E_t[e^{-rT}S(T)],$$

or

$$S(t) = E_t[e^{-r(T-t)}S(T)],$$

so

$$e^{-r(T-t)}E_t\left[S(T) - K\right] = e^{-r(T-t)}E_t[S(T)] - e^{-r(T-t)}E_t[K] = S(t) - Ke^{-r(T-t)}.$$

(b) We should differentiate to obtain

$$V_t = -rKe^{-r(T-t)}, \; V_S = 1, \quad V_{SS} = 0.$$

Then

$$V_t + rSV_S + \frac{1}{2}\sigma^2 S^2 V_{SS} = -rKe^{-r(T-t)} + rS + 0 = r\left(S - Ke^{-r(T-t)}\right) = rV.$$

Also,

$$V(T, S) = S - K$$

so the terminal condition is satisfied.

2. We should note that $\Phi(-x) = 1 - \Phi(x)$.

3. (a)

$$\begin{aligned}
dX(t) &= \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))\,dt + A(t)\Delta(t)S(t)\,dt \\
&= \Delta(t)S(t)\left(\alpha dt + \sigma\, dW(t) - A(t)dt\right) + r(X(t) - \Delta(t)S(t))\,dt + A(t)\Delta(t)S(t)\,dt \\
&= \Delta(t)\sigma S(t)\left(dW(t) + \frac{\alpha - r}{\sigma}dt\right) \\
&= \Delta(t)\sigma S(t)d\tilde{W}(t)
\end{aligned}$$

(b) Because we can replicate the payoff of the derivative using a portfolio as $X(t)$, and the portfolio value (as $r = 0$ here) is a martingale under the risk-neutral measure,

$$V(t) = X(t) = E_t[X(T)] = E_t[V(T)].$$

(c) We can solve the SDE for $S(t)$:

$$\frac{dS(t)}{S(t)} = (\alpha - q)dt + \sigma\, dW(t) = -qdt + \sigma\, dW(t)$$

to obtain

$$S(t) = S(0) \exp\left\{(-q - \frac{1}{2}\sigma^2)t + \sigma W(t)\right\}$$

So

$$\log\left(\frac{S(t)}{S(0)}\right) = -\left(q + \frac{1}{2}\sigma^2\right)t + \sigma W(t)$$

(d) To derive the PDE for the price function, we note from part (b) that V is a martingale under the risk-neutral measure, which means that dV should have only $d\tilde{W}$ terms. Using Itô's formula,

$$\begin{aligned} dV &= V_t\, dt + V_S\, dS + \frac{1}{2}V_{SS}(dS)^2 \\ &= V_t\, dt + V_S\, S\,(\alpha\, dt + \sigma\, dW(t) - q\, dt) + \frac{1}{2}\sigma^2 S^2 V_{SS}\, dt \\ &= \left(V_t - qSV_S + \frac{1}{2}\sigma^2 S^2 V_{SS}\right) dt + \sigma d\left(W(t) + \frac{\alpha t}{\sigma}\right) \\ &= \left(V_t - qSV_S + \frac{1}{2}\sigma^2 S^2 V_{SS}\right) dt + \sigma d\tilde{W}(t) \end{aligned}$$

In order that V is a martingale under the risk-neutral measure, we must have

$$V_t - qSV_S + \frac{1}{2}\sigma^2 S^2 V_{SS} = 0.$$

4. (a) We should modify the formulas in the Black-Scholes formula to have

$$\begin{aligned} d_1 &= \log\left(\frac{S(0)e^{-qT}}{K}\right) + (r + \frac{1}{2}\sigma^2)T \\ &= \log\left(\frac{S(0)}{K}\right) + (r - q + \frac{1}{2}\sigma^2)T \end{aligned}$$

and similar for d_2 so

$$C = S(0)e^{-qT}\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

(b) Notice that the combined payoff of a long position in call and a short position in put is $S(T) - K$. In order to get this payoff, we can take a long position in the stock and borrow some money at the risk-less rate. The amount to put in the stock in order to get one share of the stock at T is $S(0)e^{-qT}$, which allows you to buy only less than one share, but with reinvestment from the dividends, this position will grow to just one share at time T. So we have

$$S(0)e^{-qT} - Ke^{-rT} = c - p$$