Homework Assignment 5, Math 5765, due March 20

1. A forward contract on a non-dividend-paying stock is a derivative of the stock, with payoff $S(T) - K$ at time T, where K is the delivery price specified in the contract.

   (a) Use the risk-neutral valuation to show that the value of the contract at time $t < T$ is
   $$V(t) = e^{-r(T-t)} E_t[S(T) - K] = S(t) - Ke^{-r(T-t)}$$

   (b) Verify that $V(t, S) = S - Ke^{-r(T-t)}$ satisfies the Black-Scholes PDE, with the terminal condition $V(T, S) = S - K$.

2. The put-call parity states that for a European call c and a European put p with the same expiration T and strike K, we must have
   $$c(t, S(t)) - p(t, S(t)) = S(t) - K$$
   Use this condition and the Black-Scholes formula for the call to derive the Black-Scholes formula for the put:
   $$p(t, S(t)) = Ke^{-r(T-t)}\Phi(-d_2) - S(t)\Phi(-d_1)$$

3. This problem is intended to cover the material about stocks with continuously paying dividend and their options. Assume that the stock pays dividends continuously over time at a rate $A(t)$ per unit time, where $A > 0$ is a known function of t, the stock price will follow the process
   $$\frac{dS(t)}{S(t)} = \alpha\, dt + \sigma\, dW(t) - A(t)\, dt$$
   Suppose you manage a portfolio that consists of $\Delta(t)$ shares of the stock, and the rest in a money market account, then there will be three different sources in the change of the portfolio value$X(t)$: (1) the stock values, (2) the dividends received, and (3) the money market interest payments. For convenience, we assume that the money market account earns zero interest rate.

   (a) Show that
   $$dX(t) = \Delta(t)\sigma S(t)\, d\tilde{W}(t)$$
   for any $\Delta(t)$ that is known at time t, where
   $$\tilde{W}(t) = W(t) + \frac{\alpha t}{\sigma}$$

(b) Show that the price $V(t)$ of any derivative on $S(t)$ that pays $V(T)$ at time T satisfies

$$V(t) = \tilde{\mathbb{E}}_t\left[V(T)\right], \quad 0 \leq t \leq T.$$

(c) Given constant volatility σ, constant interest rate $r = 0$, and constant dividend rate $A(t) = q$, show that

$$\log\left(S(t)/S(0)\right) \sim N\left(-(q + \frac{1}{2}\sigma^2)t, \sigma^2 t\right).$$

(d) Show that the PDE satisfied by $V(t, S)$, the derivative price at t if the stock price is S, is

$$V_t - qSV_S + \frac{1}{2}\sigma^2 S^2 V_{SS} = 0.$$

4. For a stock that pays a continuous dividend yield at a constant rate q, we can modify the Black-Scholes formula for non-dividend-paying stocks to obtain option prices for this stock, based on the following argument: if, with a continuous dividend yield of q, the stock price grows from $S(0)$ to $S(T)$, then in the absence of dividends it would grow from $S(0)$ to $S(T)e^{qT}$. This means that in the absence of dividends it would grow from $S(0)e^{-qT}$ at time zero to $S(T)$ at time T.

(a) By replacing $S(0)$ by $S(0)e^{-qT}$ in the Black-Scholes formulas, obtain the Black-Scholes formula for a European call on a stock that pays a continuous dividend yield at rate q;

(b) Show that the put-call parity in this case is

$$c + Ke^{-rT} = p + S(0)e^{-qT}$$

(c) Pick a stock and a pair of call and put, see for what value of q the above condition is satisfied. This value q is called the implied yield or implied dividend rate of the stock.