Solutions to Homework Assignment 3

1. We should just recognize that $W(t+\Delta t) - W(t)$ is a normal random variable with mean 0 and variance Δt, and the definition of the kurtosis.

2. (a) It is straightforward to see

$$\frac{S(t_{j+1})}{S(t_j)} = \exp\left((r - \frac{1}{2}\sigma^2)\Delta t + \sigma \Delta W_j\right),$$

so

$$\begin{aligned}
\sum_{j=0}^{n-1}\left(\log\frac{S(t_{j+1})}{S(t_j)}\right)^2 &= \sum_{j=0}^{n-1}\left((r - \frac{1}{2}\sigma^2)\Delta t + \sigma\Delta W_j\right)^2 \\
&= \sigma^2\sum_{j=0}^{n-1}(\Delta W_j)^2 + (r - \frac{1}{2}\sigma^2)^2\sum_{j=0}^{n-1}(\Delta t)^2 + \\
&\quad 2\sigma(r - \frac{1}{2}\sigma^2)\sum_{j=0}^{n-1}(\Delta W_j)\cdot\Delta t
\end{aligned}$$

(b) As $\Delta t \to 0$,

$$\sum_{j=0}^{n-1}(\Delta W_j)^2 \to \sum_{j=0}^{n-1}\Delta t = T,$$

$$\sum_{j=0}^{n-1}(\Delta t)^2 = \Delta t\sum_{j=0}^{n-1}\Delta t = T\Delta t \to 0,$$

$$\sum_{j=0}^{n-1}\Delta W_j\cdot\Delta t = \frac{T}{n}\sum_{j=0}^{n-1}\Delta W_j \to 0.$$

The last part is based on the law of large numbers. After adding up these three parts, we have the limit $\sigma^2 T$.

(c)

$$\sum_{j=0}^{n-1} Y_j = (r - \frac{1}{2}\sigma^2)T + \sigma W(T) = \log\frac{S(T)}{S(0)},$$

so the term that should be subtracted from the sum is

$$\frac{1}{n}\left(\log\frac{S(T)}{S(0)}\right)^2,$$

which can be easily implemented in any estimate. To answer the question when it is justified to leave out this correction term, we note

$$\frac{1}{n}\left(\sum_{j=0}^{n-1} Y_j\right)^2 = \frac{1}{n}\left(\alpha^2 T^2 + 2\alpha\sigma T^{3/2} Z + \sigma^2 T Z^2\right)$$

where Z is a standard normal random variable and $\alpha = r - \sigma^2/2$. The mean of the sum is

$$\frac{1}{n}\left(\alpha^2 T^2 + \sigma^2 T\right),$$

and the variance of the sum is

$$\frac{1}{n^2}\left(4\alpha^2\sigma^2 T^3 + 2\sigma^4 T^2\right).$$

We claim that as n becomes very large, both the mean and the variance of this correction term approach zero, therefore it is justified to drop that correction term when n is very large, or Δt approaches zero.

(d) If r and σ are both time dependent, we need to adjust the model to

$$S(t) = S(0)\exp\left(\int_0^t \left(r(u) - \frac{1}{2}\sigma^2(u)\right) du + \int_0^t \sigma(u) dW(u)\right).$$

The sum in question becomes

$$\begin{aligned}
&\sum_{j=0}^{n-1}\left(\log\frac{S(t_{j+1})}{S(t_j)}\right)^2 \\
=\ &\sum_{j=0}^{n-1}\left((r_j - \frac{1}{2}\sigma_j^2)\Delta t + \sigma_j \Delta W_j\right)^2 \\
=\ &\sum_{j=0}^{n-1}\sigma_j^2 (\Delta W_j)^2 + \sum_{j=0}^{n-1}(r_j - \frac{1}{2}\sigma_j^2)^2(\Delta t)^2 + 2\sum_{j=0}^{n-1}\sigma_j(r_j - \frac{1}{2}\sigma_j^2)(\Delta W_j)\cdot\Delta t
\end{aligned}$$

The first term will go to

$$\int_0^T \sigma^2(t) dt$$

and the other two will also approach zero, as long as $r(t)$ and $\sigma(t)$ satisfy some boundedness conditions.

3. Let $f(t, x) = \log x$,

$$\begin{aligned}
d\log S &= \frac{1}{S} dS + \frac{1}{2}\left(-\frac{1}{S^2}\right)(dS)^2 \\
&= \alpha\, dt + \sigma\, dW - \frac{1}{2}\sigma^2\, dt \\
&= \left(\alpha - \frac{1}{2}\sigma^2\right) dt + \sigma dW
\end{aligned}$$