Homework Assignment No. 3, Math 5765, due Feb. 28 at 5 p.m.

1. The kurtosis of a random variable is the ratio of its fourth central moment to the square of its variance. For a normal random variable, the kurtosis is 3. One way to calculate this is to use the characteristic function $\phi(u) = E[e^{u(X-\mu)}]$ and note that for a normal random variable $X \sim N(0, \sigma^2)$, $\phi(u) = e^{\frac{1}{2}u^2\sigma^2}$. The 4th moment is therefore calculated as $\phi^{(4)}(0)$. Use the above results to show
$$E[(W(t+\Delta t) - W(t))^4] = 3(\Delta t)^2$$

2. Assume that the stock price follows the process,
$$S(t) = S(0)\exp\left((r - \frac{1}{2}\sigma^2)t + \sigma W(t)\right),$$
where the volatility σ and the interest rate r are taken as constants. You may want to estimate the realized variance $\int_0^T V(t)\,dt$ over $[0, T]$, using the real data $S(t_j), j = 0, 1, 2, \ldots, n$, at $t_j = j\Delta t$, where $\Delta t = T/n$. The following steps describe the procedure to estimate σ.

   (a) Show that
$$\begin{aligned}\sum_{j=0}^{n-1}\left(\log\frac{S(t_{j+1})}{S(t_j)}\right)^2 &= \sigma^2\sum_{j=0}^{n-1}(\Delta W_j)^2 + (r - \frac{1}{2}\sigma^2)^2\sum_{j=0}^{n-1}(\Delta t)^2 + \\ &\quad 2\sigma(r - \frac{1}{2}\sigma^2)\sum_{j=0}^{n-1}(\Delta W_j)\cdot\Delta t\end{aligned}$$
   where $\Delta W_j = W(t_{j+1}) - W(t_j)$;

   (b) Justify for each term to show that as $\Delta t \to 0$, the above sum converges to $\sigma^2 T$;

   (c) If we denote $Y_j = \log(S(t_{j+1})/S(t_j))$ and assume $Y_j, j = 0, 1, \ldots$ are i.i.d.'s, the variance of Y_j should be estimated based on
$$\sum_{j=0}^{n-1}(Y_j - \bar{Y})^2 = \sum_{j=0}^{n-1}Y_j^2 - \frac{1}{n}\left(\sum_{j=0}^{n-1}Y_j\right)^2$$
   where the second term on the right-hand-side is an adjustment. When do you think this adjustment is needed?

   (d) Now assume that σ is time-dependent but still deterministic, how does the formula change in part (a)? Does it matter if r is also time-dependent?

3. The generalized geometric Brownian motion equation for a stock price $S(t)$ is

$$\frac{dS(t)}{S(t)} = \alpha(t)\,dt + \sigma(t)\,dW(t)$$

Using Itô's formula to compute $d\log S(t)$. Simplify so that it does not involve $S(t)$. Then integrate the formula you obtained and exponentiate to arrive at a formula for $S(t)$.