Some Notes/hints for Final Project

1 Problem 3(c)

This is a very good application of the approximation theory. We know that if two portfolios have the same payoff function, then they should have the same price, otherwise there will be arbitrage opportunities. With this principle, if two portfolios have payoff functions very close to each other, their prices should also be close, which opens a door for all the approximations. We note that each call or put has a piecewise linear function as its payoff, then the sum of a collection of them should also be a piecewise function, just with more pieces. The final question is whether the payoff function $\log S$ can be approximated by a piecewise linear function, and from a geometric point of view we can just pick some points on the curve and connect them by straight lines. These straight lines form the piecewise linear function and you are now asked to write out that piecewise linear function in terms of a sum of call/put option payoffs.

2 Problem 6

Notice that in Merton's model formula on page 7 of the lecture notes, the mean of $\log Y_j$ is assumed to be zero.

3 Problem 8

3.1 Transforming random variables with known distribution

Suppose that a continuous random variable X has a CDF (cumulative distribution function)

$$F(x) = P\left\{X \leq x\right\},$$

which we assume to be a strictly increasing function (for our convenience) defined on some interval in $\mathbf{R}^1$. We claim that $U = F(X)$ is a random variable with uniform distribution $U(0, 1)$. To see this, we first note that $U \in (0, 1)$ and

$$P\left\{U \leq u\right\} = P\left\{F(X) \leq u\right\} = P\left\{X \leq F^{-1}(u)\right\} = F(F^{-1}(u)) = u$$

Here $F^{-1}(x)$ is the inverse function of F, such that $y = F^{-1}(x)$ implies $F(y) = x$. This relation confirms that U is a uniform random variable between 0 and 1.

3.2 Equivalent events

In the Gaussian copula function, the event in question can be described by three random variables: the default time τ, a uniformly distributed U, and a normal random variable X, and they are related through

$$U = F(\tau) = N(X),$$

where F is the CDF of the default time τ and N (sometimes we use Φ) is the CDF of standard normal. The following events are therefore equivalent:

$$\tau > t,$$

$$U > F(t),$$

and

$$X > N^{-1}(F(t)).$$

Finally, the probability we ask is

$$\begin{aligned} & P(\tau^* \leq t) \\ = \; & 1 - P(\tau^* > t) \\ = \; & 1 - P(\tau_1 > t, \tau_2 > t) \\ = \; & 1 - P\left(X > N^{-1}(Q_1(t)), Y > N^{-1}(Q_2(t))\right) \end{aligned}$$