

Summary of Math 5760, Fall 2010

December 13, 2010

In this first installment of the mathematical finance series, we introduced some of the fundamental ideas and methodology of financial derivative pricing, all through the deceptively simple binomial model. It is the richness and the ease of implementation that made the binomial model such a popular approach, both for practitioners and researchers. In the following we list the main ideas contained in this semester, all explained in the context of the binomial model.

1 Probability Preliminaries

Even though Math 5010 or equivalent is a prerequisite for this course, it is still necessary to review some fundamental concepts in probability theory. The most important but rather challenging notion used here is conditional expectation as a random variable. We want to emphasize the random variable aspect since the term can be mistakenly thought to be just a deterministic value (from the word *expectation*). As we explained in a previous note, a description of successive stock price distributions for different times is not very useful for us to follow stock price movements. What we need is a description of the transition of stock prices. The outlook for tomorrow when starting with a price of \$10 is very different from starting with a \$100 price.

Two important concepts we introduced, in conjunction with Math 5040, and need in building and analyzing our stock price models, are Markov process and martingale. Both are heavily tied to the conditional expectation framework. First of all, a Markov process is probably the simplest model for a sequence of random variables indexed by time, except for a set of completely independent random variables. This is why we often use it as the first choice to model a time related phenomenon - the alternatives are just too difficult to handle. Surprisingly that first choice works out quite well in many, if not all, financial applications. For those of you taking Math 5040 concurrently, you will see the huge amount of results developed for Markov chains and these results are very much relevant to modeling financial markets.

2 Setting of the Binomial Model - Forward Iteration

In a binomial model, there are two aspects we need to specify: the probabilities of going up/down \tilde{p} and \tilde{q} (it helps tremendously with only two possibilities, thus the name binomial), and the increase/decrease the price experiences when it goes up/down, respectively. By specifying these two aspects, you have created a binomial model. These parameters are chosen so the price movements for the next time step will have the required mean (related to the trend of the stock) and variance (related to the volatility of the stock). The variance information has not much ambiguity: the volatility of the stock is certainly observable, while the stock trend is anyone's guess. For this we need the no-arbitrage theory to justify the use of risk-less interest rate.

3 Using a Binomial Model - Backward Iteration

Once we have the binomial model for a particular stock setup, we can use it to price any derivative that is written on the stock. The procedure has two key components: calculate the payoff at expiration $t = T$, or $n = N$ in the discrete time model, and propagate the value backward in time, based on expectation and discount:

$$V_n(\omega_1\omega_2\cdots\omega_n) = \frac{1}{1+r\Delta t} (\tilde{p}V_{n+1}(\omega_1\omega_2\cdots\omega_nH) + \tilde{q}V_{n+1}(\omega_1\omega_2\cdots\omega_nT))$$

This procedure not only gives you the price of the derivative at time 0, it also tells you the price at a later time for each particular state.

4 State Prices, Change of Measure, and Martingale

The state price concept is a link between the mathematical framework based on martingale theory and classic economic theories such as utility function and indifference curves. It incorporates two factors: the discount factor, and the probability factor, albeit in the risk-neutral world. Intuitively speaking, it measures how much an investor in the risk-neutral world is willing to bet on this state. You can argue that different investors have different taste for risk: some are risk-averse and some are risk-seeking, therefore this bet is not well defined. This is precisely the point of risk-neutral world: every investor has the same attitude towards risks, so the amount they are willing to bet on one state is the same, which defines the state price.

To justify the procedures in Sections 2 and 3, we showed the following:

- No arbitrage in the market implies the existence of a risk-neutral measure under which all securities (the stock and its derivatives, and the money market account) are martingales;
- This risk-neutral measure is derived from the condition that the stock price under which is a martingale, this is the forward induction step;

- Once the risk-neutral measure is obtained, it can be used to price derivatives written on that stock, this is the backward induction step.

The difference between the risk-neutral measure and the actual world measure is bridged by a change of measure and the calculations are facilitated through the Radon-Nikodým process and derivative. It is quite striking to note that the volatility of the stock is preserved after the change of measure and the insight will be provided in the study of Girsanov's theorem next semester.

5 American Style

In the binomial model setting, the American style derivative pricing is naturally extended by incorporating an extra step that compares the discounted expectation of future payoff with the immediate payoff. Even if the derivative is path-dependent, the conceptual procedure is still quite simple (the computational cost is another matter). A deep question to test your understanding of the subject is the argument/proof that the American call on a stock that pays no dividend is the same as the European counterpart.

6 Random Walk, a Preamble to Brownian Motion

This is a preparation for the next semester, when Brownian motion will be introduced as the basic starting point for continuous time models. The nice thing about random walk is that we know a lot about the crossing behavior and the wonderful diffusion theory comes into play. At this point, we can present the basic Black-Scholes model in the discrete time version:

$$\frac{\Delta S_n}{S_n} = r\Delta t + \sigma\sqrt{\Delta t}X_n,$$

where X_n is the step in the standard symmetric random walk. This model has an obvious interpretation: the return rate over Δt has two components: a deterministic part that is proportional to the trend and time elapsed, and a random component with a variance $\sigma^2\Delta t$.

The reflection principle is just a bonus that allows us to analytically compute many exotic option prices.

7 Interest Rates and Basic Bond Mathematics

Interest rate modeling is difficult due to the fact that at one time there are many rates (with different maturities), and they fluctuate in time in their own ways. We will need to model a vector process but cannot ignore the inner relations. If both the 2-year and 3-year rates went up, there is no easy justification for a case where the 2.5-year rate went down. In this semester, we lightly touched the modeling issue by looking at the grand

simplification: the short rate models. Ideally, we would like to model $B_{n,m}$, the time t_n discount factor for \$1 paid at time t_m . There are two time related indices: the time of observation and the maturity. But the short rate model only follows R_n , the short rate prevailing for period $[t_n, t_{n+1}]$. If one particular R_n goes up, all $B_{n,m}$ would go down for all $m > n$. To allow situations where some bond prices go up and some others go down at the same time, we would need more sophisticated model than this.

The advantage of the short rate model is that what we learned in stock price models can be directly transcribed to interest rates. The forward and backward iterations are used in very similar ways.

We also studied some fundamental bond mathematics. It is important to understand how the yield is defined through the bond price. We should know that coupons are not the interest we earn from the investment. You can find a bond with large coupon rates but the bond price may fall in time, so the coupons you receive along the way cannot be justified as interests since you are losing money on the bond price itself.