Summary of Chapter 1

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In this first chapter, we introduced several basic concepts in derivative pricing based on the no-arbitrage principle, and developed the backbone binomial tree model to price fairly general derivatives. Contained in this short chapter are the discussion of classic options and the main idea in pricing that leads to the basic binomial model which is still widely used in practice. The most surprising but key result is that the expected growth rate (μ) of the underlying stock does not enter the pricing of derivatives written on it. As it turns out, a crucial aspect of the stock that matters more than any other, to certain extent in derivative pricing, is that elusive quantity called volatility, or σ, of the stock. It is elusive because it cannot be exactly measured, but powerful because there is no denying that it is there and driving the market.

1 Trading securities

The subject in finance that we are exploring is financial market, the mechanism that facilitates and controls the flow of capital in our economy. A reliable and efficient financial market is crucial to the operation of the economy, and the emergence of derivatives has made the market a quite different mechanism that requires all the mathematical tools to analyze. The sheer size of the derivative market and the ubiquitous presence of embedded options have forced the society to face these constantly evolving securities, and the studying of derivative securities is required both at an individual level if you are interested in a career in that field, and at a public level concerning policies that affect everyone. One aspect of modern financial market that we should recognize is that trading is taking place at every moment, and by trading there is not necessarily any asset physically changing hands. Everything is done electronically with a few clicks, or automatically according to some programs developed by some quants (very likely former mathematicians). By financial security we refer to some asset that can be bought or sold at any time, with the value determined by the market, if it is efficient. One useful feature when dealing with trading securities that may be uncomfortable for some is the practice of short selling, that is selling some security that you don't actually own. Since there is no actual delivery, short

selling is nothing but an accounting issue: someone's loss is your gain and someone's gain is your loss. If it is during a rather quiet period (i.e. no unusual events) and the security is not under particular stress, the short sell can go quite smoothly. The proliferation of derivative securities can be contributed to the widespread practice of short selling. The other related question in trading regarding the security is its liquidity: how easy you can buy or sell the security, mostly reflected in its trading volume. The attraction of a liquid security to the investors is that it can be easily got rid of if the security became a liability. It should be pointed out that the liquidity issue is not included in the models we discuss, and this can be viewed as one defect of the models that can explain the gap between theory and reality.

When it comes to derivative securities, especially those with fine prints that nobody seems to understand, we also wonder the purpose of such complicated products. It is true that they sometimes do more harm than good, but in some instances they actually serve a purpose for investors with a particular need. On the other hand, the seller (banks) of the derivative then takes on a substantial risk, as we can see from various examples. The fact that it is still a huge business for Wall Street firms is due to the discovery of replicating portfolios, which suggested that it is possible to write a derivative contract and then protect yourself by doing the right hedge, that is to invest in other securities that would eliminate the risk from the derivative you sold. The seller (writer) of the derivative is often rewarded with a higher selling price than the replicating cost. This would have been the case if the securities had been related to each other as described in the model. In other words, the success depends on whether the hedge actually does its job as promised. Markets are always unpredictable but we often notice some relations among certain securities. In the case of derivatives and their underlyings, sometimes that relationship could be established through mathematical models. This leads to an industry where analysts with strong mathematical skills try to design, create, and hedge certain financial products to cater to various needs of the investors. It is the trustworthy of various sophisticated models that the derivative trading systems now depend on. Our mission here is to look into this world of modeling and judge by ourselves whether we should accept these claims, and what efforts we should make to utilize the resources, and limit our losses in the event of another financial crisis that seems unavoidable in the future.

2 Notations

In probability theory, upper case letters are usually reserved for random variables, and lower case letters are used for functions or just variables. When we see notations like X and x, the former is a random variable, sometimes expressed as $X(\omega)$ to emphasize this fact that it depends on this random factor ω, while the latter denotes a real value, often used as an argument in a function. The expression $P[X = x]$ actually refers to a function of x: the probability of X assuming a given value x. We encounter notations like V_n and

v_n quite often in this text. The following expression is a typical example:

$$V_n(\omega_1\omega_2\ldots\omega_n) = v_n(S_n(\omega_1\omega_2\ldots\omega_n)). \tag{1}$$

Here the left hand side suggests that V_n is a random variable, indexed by time n, and depends on the first n tosses. The right hand side is a deterministic function v_n (also indexed by n) evaluated at a random variable S_n, which depends on the first n tosses. One illustration is $v_n(x) = x^2$, which leads to $V_n = S_n^2$.

3 No-arbitrage principle

There are several pieces in this puzzle of derivative pricing: a replicating portfolio, no-arbitrage principle and no-arbitrage price, and risk-neutral measure. We summarize these pieces here once again, and hope that it will help us to put them together by the end of this discussion.

1. *Replicating portfolio*

   If there is certain trading *strategy, or rule*, that tells us what to do under various market conditions, like determining what securities to buy or sell and in what quantities, so that the final portfolio value matches another security (such as a call option on the stock) in all possible outcomes, then we say that we have a replicating portfolio that will replicate the other security in question. The other security we try to replicate is often a derivative security. In the one-period binomial model, we can use the stock and the money market to replicate a stock option by taking certain shares in each and wait for the outcome. All we need to know is the list of all possible scenarios for the stock price, but not how likely each scenario turns up. In a multi-period model, a static hedge will not be sufficient, instead we need to rebalance the portfolio after each time step, which is called dynamic hedging. What matters is the following: a). the fact that the value of the replicating portfolio in each outcome matches the payoff of the derivative security in the corresponding outcome; b). there should be no new money injected to the portfolio: you can only transfer money from one account to another. By creating a replicating portfolio, we are making a statement that even though we cannot predict the stock price on a future date, we can be completely sure that our portfolio will *match* exactly the derivative payoff at the expiration time T, in all circumstances.

2. *No-arbitrage principle and no-arbitrage price*

   The fact that the replicating portfolio does exactly the same job as the derivative dictates that the price of the portfolio should match the derivative price at any time before the expiration, that is $V_n(\omega_1\omega_2\ldots\omega_n) = X_n(\omega_1\omega_2\ldots\omega_n)$ for $n = 0, 1, \ldots, N$, and all possible combinations of $\omega_1\omega_2\ldots\omega_n$. This is based on the no-arbitrage principle: arbitrage opportunities will disappear as soon as they appear. Since the

stock price and the money market value are observable, we are then led to a deriva-tive price determined by the stock prices up to that time, *if* we know the respective shares of stock and money market to own, and this information is contained in the hedge ratio Δ_n.

The derivative price based on the above portfolio construction and the no-arbitrage principle is called the no-arbitrage price. It can be viewed as the only price that would avoid arbitrage, therefore a fair price for trading at that moment.

3. *Risk-neutral measure and martingale approach*

The martingale approach is suggested by the following observation: the value of the replicating portfolio follows the dynamics

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n), \quad n = 0, 1, \ldots, N-1. \tag{2}$$

If we introduce the discounted stock price $\tilde{S}_n = S_n/(1+r)^n$, and the discounted portfolio value $\tilde{X}_n = X_n/(1+r)^n$, the above relation becomes

$$\tilde{X}_{n+1} - \tilde{X}_n = \Delta_n(\tilde{S}_{n+1} - \tilde{S}_n). \tag{3}$$

If the discounted stock price change has mean zero, the discount portfolio value change has the same feature. In probability theory, the concept of martingale is introduced to describe such behavior, and this approach in derivative pricing is often call the martingale approach. It follows that if $\{\tilde{S}_n\}$ is a martingale, so is $\{\tilde{X}_n\}$, so is $\{\tilde{V}_n = V_n/(1+r)^n\}$. The martingale theory will then provide many beautiful results for the derivative price V_n.

How do we know that discounted stock price change has mean zero? Even discounted stock prices are expected to grow, or else nobody would be interested in buying stocks since there are inherent risks involved. However, we know that the growth rate is affected by the probabilities of the possible scenarios, and if we change the probabilities we can force the expected growth rate to be zero. By choosing this special probability measure, we send ourselves to a very strange world. In that world, as long as the expected growth is the same, people do not care about the level of the risk a particular security carries. In the real world, if a stock is expected to grow at the same rate as the risk-free bank deposit, most people will avoid it since the stock carries a risk by its nature. To most risk-averse investors, they would only be interested in a risky stock if it offers something extra in compensation for the risk involved (some may never be interested), such as a much higher expected return compared to that of a risk-free bank deposit. In that strange world, however, people neither tried to avoid risk, nor to seek risk, all they care about is that the expected return matches the bank risk-free rate. That world is called a *risk-neutral* world and the probability measure in that world is called the risk-neutral measure.

Once we established that $\tilde{V}_n$ is a martingale under the risk-neutral measure (with expectation denoted by $\tilde{E}$), we have

$$\tilde{V}_n = \tilde{E}_n[\tilde{V}_{n+1}]. \tag{4}$$

This formula is extremely powerful in that it gives us a backward induction procedure to price any derivative, at any time before the expiration. It is the fundamental framework of derivative pricing.