

Solutions for Assignment No. 8

Problem 4.2

We consider one step at a time. At time zero, the agent has a loan in the amount \$1.36 and owns a put. If he doesn't hedge, in the case the first toss is a head, he would not have enough amount in his asset (the put) to pay off the loan (which grows to \$1.7); on the other hand, he would have excess amount in his put to pay off the loan if the coin toss is a tail. What he should do is to buy some stock to offset the loss/gain in the put. Exactly how many shares he should buy is determined by the hedge ratio

$$\Delta_0 = \frac{0.4 - 3}{8 - 2} = -0.4333.$$

Note Δ_0 is the number of shares you need to buy to replicate the put. Therefore if the agent wants to hedge the put, he should buy 0.4333 shares of the stock, which costs him \$1.7333 and the amount needs to be financed (meaning more loan). The total loan is now $\$1.36 + \$1.7333 = \$3.0933$. Now suppose the first toss is a head, the agent owns a put (valued at \$0.40) and 0.4333 shares of the stock (valued at $0.4333 \times \$8 \approx \3.4667) so his total asset is approximately \$3.8667, which covers the load with interest $1.25 \times \$3.0933$ exactly.

If the first toss is a tail, the agent should exercise the put and take \$3 from the payoff, and his stock value is $0.4333 \times \$2 \approx \0.8667 . With these funds he can also cover the loan with interest \$3.8667. At this point, the game is over as the put has been exercised and the stock position should also be closed out to pay out the remaining loan.

In the case the first toss is head and the agent wants to continue on to time two, if the first toss is head, the new hedge ratio is

$$\Delta_1(H) = \frac{0 - 1}{16 - 4} = -\frac{1}{12},$$

so the agent should sell $0.4333 - 0.0833 = 0.35$ shares and use the proceed to pay off part of the loan. By the time two, regardless of the outcome of the second toss, he should have enough (put + stock) to cover the remaining loan.

Problem 4.5

There are a total of 26 stopping times and each of them can be listed in the form

$$[\tau(HH), \tau(HT), \tau(TH), \tau(TT)].$$

- Constant times:

$$[0, 0, 0, 0], [1, 1, 1, 1], [2, 2, 2, 2], [\infty, \infty, \infty, \infty];$$

- Stopping at 1 at least for some paths:

$$\begin{aligned} &[1, 1, 2, 2], [1, 1, \infty, \infty], [1, 1, 2, \infty], [1, 1, \infty, 2], \\ &[2, 2, 1, 1], [\infty, \infty, 1, 1], [\infty, 2, 1, 1], [2, \infty, 1, 1]; \end{aligned}$$

- Stopping at 2 at the earliest:

$$\begin{aligned}
& [2, 2, 2, \infty], [2, 2, \infty, 2], [2, \infty, 2, 2], [\infty, 2, 2, 2], \\
& [2, 2, \infty, \infty], [2, \infty, 2, \infty,], [\infty, 2, \infty, 2], [\infty, \infty, 2, 2], [2, \infty, \infty, 2], [\infty, 2, 2, \infty], \\
& [\infty, \infty, \infty, 2], [\infty, \infty, 2, \infty,], [\infty, 2, \infty, \infty], [2, \infty, \infty, \infty].
\end{aligned}$$

In order to maximize the expectation in equation (4.4.5), we need to eliminate those stopping times that can lead to exercise when the option is out of the money (corresponding to irrational decisions that would work against holder's interest). Obviously, the holder should never exercise in the following situations: the first toss is head (corresponding to $\tau(HH) = \tau(HT) = 1$), the first and the second tosses are both head ($\tau(HH) = 2$). Going through the above list and eliminate those with $\tau(HH) = \tau(HT) = 1$, or $\tau(HH) = 2$, we are left with

- Constants:

$$[0, 0, 0, 0], [\infty, \infty, \infty, \infty];$$

- Stopping at 1 at least for some paths:

$$[\infty, \infty, 1, 1], [\infty, 2, 1, 1];$$

- Stopping at 2 for the earliest:

$$\begin{aligned}
& [\infty, 2, 2, 2], [\infty, 2, \infty, 2], [\infty, \infty, 2, 2], [\infty, 2, 2, \infty], [\infty, \infty, \infty, 2], \\
& [\infty, \infty, 2, \infty,], [\infty, 2, \infty, \infty].
\end{aligned}$$

These are the possible candidates for our optimization problem. Computing the expectations for these 11 cases is straightforward. Notice the simplification when $\tau = \infty$ for some path, which means that there is no contribution (zero cash flow) to the expectation from that path.

Problem 4.6

- When we take an expected value that involves a stopping time, a common trick is to construct a stopped process, since S_τ is actually not a process. We first define the discounted stock price process

$$\tilde{S}_n = \frac{S_n}{(1+r)^n},$$

which is a martingale under the risk-neutral measure. Then the stopped process $\tilde{S}_{n \wedge \tau}$ is also a martingale for any stopping time τ . In particular, for those $\tau \in \mathcal{S}_0$ and $\tau \leq N$, we have

$$S_0 = \frac{S_{0 \wedge \tau}}{(1+r)^{0 \wedge \tau}} = \tilde{S}_{0 \wedge \tau} = \tilde{\mathbb{E}}[\tilde{S}_{N \wedge \tau}] = \tilde{\mathbb{E}}\left[\frac{S_{N \wedge \tau}}{(1+r)^{N \wedge \tau}}\right] = \tilde{\mathbb{E}}\left[\frac{S_\tau}{(1+r)^\tau}\right]$$

With the above observation we can establish part (i) by noticing

$$\begin{aligned}\tilde{\mathbb{E}} \left[\frac{G_\tau}{(1+r)^\tau} \right] &= \tilde{\mathbb{E}} \left[\frac{K - S_\tau}{(1+r)^\tau} \right] = \tilde{\mathbb{E}} \left[\frac{K}{(1+r)^\tau} \right] - \tilde{\mathbb{E}} \left[\frac{S_\tau}{(1+r)^\tau} \right] \\ &= K \tilde{\mathbb{E}} \left[\frac{1}{(1+r)^\tau} \right] - S_0.\end{aligned}$$

The first term on the right hand side depends on what stopping time we use. If we take the maximum over all $\tau \in \mathcal{S}_r$ and $\tau \leq N$, we find that the maximum is attained by the stopping time $\tau = 0$ since $r \geq 0$. So we have

$$V_0 = K - S_0,$$

which implies that the derivative is exercised at time zero.

There is another approach (more practical) to solve this problem by following the algorithm that is modified to accommodate this particular condition that the derivative must be exercised by N . If the derivative is held till N ,

$$V_N = G_N = K - S_N.$$

By moving backward in one time, the value of this security at $N - 1$ is

$$\begin{aligned}V_{N-1}(S_{N-1}) &= \max\{G_{N-1}, \frac{1}{1+r} (\tilde{p}V_N(uS_{N-1}) + \tilde{q}V_N(dS_{N-1}))\} \\ &= \max\{K - S_{N-1}, \frac{K}{1+r} - S_{N-1}\} \\ &= K - S_{N-1}\end{aligned}$$

Here we used the fact for the risk-neutral probabilities:

$$\tilde{p}S_{n+1}(H) + \tilde{q}S_{n+1}(T) = (1+r)S_n.$$

In general, we can show

$$V_n = K - S_n > \frac{1}{1+r} \tilde{\mathbb{E}}_n[V_{n+1}],$$

for $n = 0, 1, \dots, N - 1$. This shows that the holder should immediately exercise at n if it has not been exercised earlier. Using induction, we conclude that the derivative should be exercised at $n = 0$, resulting

$$V_0 = K - S_0.$$

(ii) Let's compare the following three portfolios:

- A: the above security, and the European call;
- B: the above security, and the American call;
- C: the American put.

We note that portfolio B would have been equivalent to C if these two securities in portfolio B had been required to be exercised at the same time, because the intrinsic values would be $K - S_n$ if $S_n < K$, and zero if $S_n \geq K$. But the actual portfolio B has the flexibility that each security can be exercised independently at any time, thus should be more valuable than C:

$$\text{Value}(B) \geq \text{Value}(C).$$

On the other hand, since the American call is the same as the European call (no dividend is mentioned in this problem),

$$\text{Value}(C) \leq \text{Value}(B) = \text{Value}(A),$$

or

$$V_0^{AP} \leq K - S_0 + V_0^{EC}.$$

(iii) The put-call parity is

$$\frac{K}{(1+r)^N} - S_0 + V_0^{EC} = V_0^{EP}.$$

Since $V_0^{EP} \leq V_0^{AP}$, we have the desired lower bound for V_0^{AP} .