Solutions for Assignment No. 7

Problem 3.5
The risk-neutral probability measure is described by

$$\tilde{\mathbb{P}}(HH) = 1/4,\ \tilde{\mathbb{P}}(HT) = 1/4,\ \tilde{\mathbb{P}}(TH) = 1/12,\ \tilde{\mathbb{P}}(TT) = 5/12.$$

(i) Radon-Nikodým derivative

$$Z(HH) = 9/16,\ Z(HT) = 9/8,\ Z(TH) = 3/8,\ Z(TT) = 15/4.$$

(ii) R-N process

$$Z_1(H) = 3/4,\ Z_1(T) = 3/2,\ Z_0 = 1.$$

(iii) Derivative prices

$$V_1(H) = 12/5,\ V_1(T) = 1/9,\ V_0 = 226/225.$$

Problem 3.6
First we show that this formula is valid for $n = N$. We use (3.3.25) directly by using $I(y) = 1/y$

$$X_N = \frac{(1+r)^N}{\lambda Z},$$

and λ is solved from (3.3.26)

$$\mathbb{E}\left[\frac{Z}{(1+r)^N}\frac{(1+r)^N}{\lambda Z}\right] = \frac{1}{\lambda} = X_0.$$

Combining these two formulas,

$$X_N = \frac{(1+r)^N}{Z}X_0 = \frac{X_0}{\zeta}.$$

Next we want to show that this formula is also valid for $n < N$. For this we need to use the martingale property under the risk-neutral measure and the above result,

$$\frac{X_n}{(1+r)^n} = \tilde{\mathbb{E}}_n\left[\frac{X_N}{(1+r)^N}\right] = \tilde{\mathbb{E}}_n\left[\frac{X_0}{\zeta(1+r)^N}\right] = X_0\tilde{\mathbb{E}}_n\left[\frac{1}{Z}\right].$$

Finally we use (3.2.5) for $Y = 1/Z$ to obtain

$$\tilde{\mathbb{E}}_n\left[\frac{1}{Z}\right] = \frac{1}{Z_n}\mathbb{E}_n\left[Z\frac{1}{Z}\right] = \frac{1}{Z_n}$$

So

$$X_n = \frac{(1+r)^n}{Z_n}X_0 = \frac{X_0}{\zeta_n}.$$