Solutions for Assignment No. 6

Problem 2.10

(i) First we calculate the conditional expectation

$$\begin{aligned}
\tilde{\mathbb{E}}_n[X_{n+1}] &= \Delta_n S_n \tilde{\mathbb{E}}_n[Y_{n+1}] + (1+r)(X_n - \Delta_n S_n) \\
&= \Delta_n S_n(\tilde{p}u + \tilde{q}d) + (1+r)(X_n - \Delta_n S_n) \\
&= (1+r)X_n + \Delta_n S_n\left(\tilde{p}u + \tilde{q}d - 1 - r\right) \\
&= (1+r)X_n.
\end{aligned}$$

This shows that $X_n/(1+r)^n$ is a martingale under the risk-neutral measure.

(ii) If we choose Δ_n to be

$$\Delta_n = \frac{V_{n+1}(H) - V_{n+1}(T)}{S_n(u-d)},$$

then we will have $X_{n+1}(H) = V_{n+1}(H)$ and $X_{n+1}(T) = V_{n+1}(T)$ (verify!). Again, we have replicated the payoff of the derivative, so we can claim $V_n = X_n$ for all coin tosses at any time n. In particular, since the discounted portfolio price $X_n/(1+r)^n$ is a martingale under the risk-neutral measure, as shown in part (i), we have the pricing formula

$$V_n = \frac{1}{1+r}\tilde{\mathbb{E}}_n[V_{n+1}].$$

(iii)

$$\begin{aligned}
\tilde{\mathbb{E}}_n[S_{n+1}] &= \tilde{\mathbb{E}}_n\left[(1-A_{n+1})Y_{n+1}S_n\right] \\
&= S_n\tilde{\mathbb{E}}_n[Y_{n+1}] - S_n\tilde{\mathbb{E}}_n[A_{n+1}Y_{n+1}] \\
&= S_n(\tilde{p}u + \tilde{q}d) - S_n\tilde{\mathbb{E}}_n[A_{n+1}Y_{n+1}] \\
&= (1+r)S_n - S_n\tilde{\mathbb{E}}_n[A_{n+1}Y_{n+1}].
\end{aligned}$$

Since $\tilde{\mathbb{E}}_n[A_{n+1}Y_{n+1}]$ in general is not zero, the discounted S_n is not a martingale under the risk-neutral measure.

However, if $A_{n+1} = \alpha$ is a constant,

$$\tilde{\mathbb{E}}_n[A_{n+1}Y_{n+1}] = \alpha\tilde{\mathbb{E}}_n[Y_{n+1}] = \alpha(1+r).$$

Therefore,

$$\tilde{\mathbb{E}}_n[S_{n+1}] = (1+r)S_n - \alpha(1+r)S_n = (1-\alpha)(1+r)S_n,$$

which implies that $S_n/\left((1-\alpha)^n(1+r)^n\right)$ is a martingale under the risk-neutral measure.

Problem 2.11

(i) The payoff of the forward contract that requires you to buy one share of the stock at time N for K dollars is $S_N - K$, and the payoff of the put with strike K at expiration is $(K - S_N)^+$. The combination is therefore

$$S_N - K + (K - S_N)^+ = \begin{cases} 0, & S_N \leq K \\ S_N - K, & S_N > K \end{cases} = (S_N - K)^+.$$

This shows $F_N + P_N = C_N$.

(ii) All three securities are derivatives of the same stock so the general pricing formula applies:

$$C_n = \tilde{\mathbb{E}}_n \left[\frac{C_N}{(1+r)^{N-n}} \right],$$

$$P_n = \tilde{\mathbb{E}}_n \left[\frac{P_N}{(1+r)^{N-n}} \right],$$

and

$$F_n = \tilde{\mathbb{E}}_n \left[\frac{F_N}{(1+r)^{N-n}} \right],$$

for $0 \leq n < N$, using part (i) and linearity of conditional expections, we have $F_n + P_n = C_n$.

(iii) We start with the payoff at time N for the forward contract

$$F_N = S_N - K,$$

and divide both sides by $(1+r)^N$, then take the expectation under the risk-neutral probability measure,

$$F_0 = \tilde{\mathbb{E}} \left[\frac{F_N}{(1+r)^N} \right] = \tilde{\mathbb{E}} \left[\frac{S_N}{(1+r)^N} \right] - \frac{K}{(1+r)^N} = S_0 - \frac{K}{(1+r)^N}.$$

(iv) Since the stock price at time zero is S_0, to buy one share you would need to borrow an extra amount $S_0 - F_0 = K/(1+r)^N$ (if that amount is negative you just deposit it in a bank rather than borrow). At time N, your stock is worth S_N and your outstanding debt is $(S_0 - F_0)(1+r)^N = K$ so the total portfolio value is $S_N - K$.

(v) If we set $F_0 = 0$, we must choose $K = (1+r)^N S_0$. With this particular choice of K, $C_0 = P_0$ based on part (ii).

(vi) Suppose we choose a value $K = (1+r)^N S_0$ such that $F_0 = 0$, there is no reason for the price of the forward contract F_n to stay zero, therefore according to part (ii) we will *not* have $C_n = P_n$ for $n > 0$ in general.