Solutions for Assignment No. 3

1. (a) By definition,

$$\begin{aligned}
\tilde{\mathbb{E}}_n\left[\frac{\Delta S_n}{S_n}\right] &= \tilde{p}(u-1)+\tilde{q}(d-1) \\
&= \tilde{p}u+\tilde{q}d-1 \\
&= \frac{ue^{r\Delta t}-ud+ud-de^{r\Delta t}}{u-d}-1 \\
&= e^{r\Delta t}-1.
\end{aligned}$$

You need to use the Taylor expansion for e^x:

$$e^x = 1+x+\frac{1}{2}x^2+\frac{1}{3!}x^3+\cdots$$

With this in mind, we have

$$\tilde{\mathbb{E}}_n\left[\frac{\Delta S_n}{S_n}\right] = (1+r\Delta t+\cdots)-1 \approx r\Delta t$$

For the variance,

$$\begin{aligned}
Var\left[\frac{\Delta S_n}{S_n}\right] &= \tilde{\mathbb{E}}_n\left[\left(\frac{\Delta S_n}{S_n}\right)^2\right]-\left(\tilde{\mathbb{E}}_n\left[\frac{\Delta S_n}{S_n}\right]\right)^2 \\
&= \tilde{p}(u-1)^2+\tilde{q}(d-1)^2-(e^{r\Delta t}-1)^2
\end{aligned}$$

Now we use

$$(u-1)^2 = \left[\left(1+\sigma\sqrt{\Delta t}+\frac{1}{2}\sigma^2\Delta t+\cdots\right)-1\right]^2 = \sigma^2\Delta t+\cdots$$

and

$$(d-1)^2 = \left[\left(1-\sigma\sqrt{\Delta t}+\frac{1}{2}\sigma^2\Delta t+\cdots\right)-1\right]^2 = \sigma^2\Delta t+\cdots$$

So the variance

$$Var\left[\frac{\Delta S_n}{S_n}\right] \approx \sigma^2\Delta t(\tilde{p}+\tilde{q})-(r\Delta t)^2 = \sigma^2\Delta t+\cdots$$

(b) The numerator of $\tilde{p}$ is

$$\begin{aligned}
e^{r\Delta t}-e^{-\sigma\sqrt{\Delta t}} &= (1+r\Delta t+\cdots)-\left(1-\sigma\sqrt{\Delta t}+\frac{1}{2}\sigma^2\Delta t+\cdots\right) \\
&= \sigma\sqrt{\Delta t}+\left(r-\frac{1}{2}\sigma^2\right)\Delta t+\cdots
\end{aligned}$$

On the other hand, the denominator

$$\begin{aligned} u - d &= \left(1 + \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2 \Delta t + \cdots\right) - \left(1 - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2 \Delta t + \cdots\right) \\ &= 2\sigma\sqrt{\Delta t} + \cdots \end{aligned}$$

So

$$\tilde{p} \approx \frac{1}{2}\left(1 + \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\sqrt{\Delta t}\right)$$

The approximation for $\tilde{q}$ is similar.

2. The daily returns of SSO are reasonably close to two times that of SPX. But the YTD return of SSO falls short of the two times target. The math related explanation can be made once we learn about Ito calculus. At this point, all we can say is that the failure is partially due to the presence of volatility. Also we should note that the volatility of the SSO is almost double that of SPX. In both cases (SSO and SPX), we should note a negative correlation between the volatility and the underlying fluctutions.