Solutions for Assignment No. 1

1. (a) From the definition of the max function, we note for all $S_T > 0$,

$$\max(S_T - K, 0) - \max(K - S_T, 0) = S_T - K.$$

The left hand side is the net payoff from a long position in call and a short position in put. The right hand side is the payoff for the forward contract. If you are long one share of the stock, and borrow an amount K that is expected to be paid back at T, the combination of the two will also give you this function as the payoff at time T.

It should be emphasized that the above is valid for any stock price $S_T > 0$.

(b) You can take a short position in the call, a long position in the put, buy one share of the stock, and borrow certain amount from the bank so that the cost to set up is zero ("$-$" sign showing amount received):

short call	-	\$	2.00
long put	+	\$	1.50
long stock	+	\$	50.00
borrow	-	\$	49.50
total cost		\$	0.00

At time T, no matter what happens, the positions will end up a net value of \$0.50, regardless of the stock price at that time. Since this portfolio has no cost in setting up, and generates a profit of \$ 0.50 with no risk at all, in a world where interest rate is zero, this would be considered an arbitrage.

(c) Now with 1% interest rate, the above portfolio will require you to return $\$49.50 \times 1.01 \approx \$50$ at time T for the \$49.50 loan. Then the net value of the portfolio will be zero, no matter what value S_T is at that time. It turns out that the interest will just eat up the little arbitrage opportunity we had in part (b).

2. There are two parts in the proof: the calculation part and the no-arbitrage argument part. For the calculation, we only need to show

$$\begin{aligned} X_1 &= \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0) \\ &= \Delta_0 \left(S_1 - (1+r)S_0\right). \end{aligned}$$

Next we argue that $0 < d < 1 + r < u$ is the no-arbitrage condition. Let us assume $\Delta_0 > 0$ for now (the same conclusion holds if $\Delta_0 < 0$ is assumed). If the toss turns up head, $X_1(H) = \Delta_0 S_0(u - 1 - r) > 0$. If the toss turns up tail, $X_1(T) = \Delta_0 S_0(d - 1 - r) < 0$. This shows that the aforementioned condition will ensure that $X_1 > 0$ corresponds to the head outcome, and $X_1 < 0$ corresponds to the tail outcome. Since both the head and the tail have positive probabilities (a model property), the probabilities for $X_1 > 0$ and $X_1 < 0$ are both positive.

If you assume $\Delta_0 < 0$, then we will have $X_1(H) < 0$ and $X_1(T) > 0$. In either case, X_1 will have different signs for different toss outcomes, which shows that $X_1 > 0$ with positive probability and $X_1 < 0$ also with positive probability.

3. The bank has a long position in the call (European), and it will benefit from an upward move in the underlying stock, but suffers a huge loss if a downward move in the underlying stock turns up. Many conservative bank managers would try to do something to limit the loss, or for this matter, completely eliminate the risk (also forgo the potential gain). This example shows how it can be done in a very naive approach: reverse engineering. As we know, the call is merely a clever combination of the stock and money market, and we recognize that the risk comes from the stock component. In the case of our example, we know that $\Delta_0 = 1/2$ is the hedge ratio, that is, for each call, you should buy 1/2 shares of the stock and borrow 0.80 from someone, with the resulting portfolio replicating the payoff of the call. As we identified the source of risk, we should be able to eliminate it by doing the opposite: sell 1/2 shares of the stock and deposit the amount received for short selling ($1/2 \times 4 = \$2$) in a money market. This takes no cost to set up and will generate
$$2 \times 1.25 - \frac{1}{2} \times 8 = -1.50$$
in the case of stock up move, and
$$2 \times 1.25 - \frac{1}{2} \times 2 = 1.50$$
in the case of stock down move. Combined with the payoff from the call (\$3 for the up move and \$0 for the down move), we end up with \$1.50 in both stock move scenarios.