Homework Assignment 3, Math 5760, due Sept. 30 at 5 pm.

1. In the multiperiod binomial model, we need to choose the parameters u and d, and the risk-neutral probabilities \tilde{p} and \tilde{q} , which are dependent on u and d, to match the following two conditions:

$$\tilde{\mathbb{E}}_n\left[\frac{\Delta S_n}{S_n}\right] = r\Delta t, \quad \tilde{Var}\left[\frac{\Delta S_n}{S_n}\right] = \tilde{\mathbb{E}}_n\left[\left(\frac{\Delta S_n}{S_n} - r\Delta t\right)^2\right] = \sigma^2\Delta t.$$

(a) Show that the following choices

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}},$$
$$\tilde{p} = \frac{e^{r\Delta t} - d}{u - d}, \quad \tilde{q} = \frac{u - e^{r\Delta t}}{u - d},$$

satisfy these two matching conditions, up to the order of $O(\Delta t)$. **Hint**: since Δt is assumed to be small and we are interested in approximations, you can ignore terms of order $(\Delta t)^2$ or higher, when you use Taylor expansions.

(b) Show that the expressions for \tilde{p} and \tilde{q} can be further approximated by

$$\tilde{p} = \frac{1}{2} \left(1 + \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right) \sqrt{\Delta t} \right), \quad \tilde{q} = \frac{1}{2} \left(1 - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right) \sqrt{\Delta t} \right)$$

- 2. Download the daily closing prices for the index S&P 500 (SPX) and the ETF ProShare Ultra S&P 500 (SSO) from finance.yahoo.com for the period beginning 1/4/2010 till now.
 - (a) Compute the daily percentage returns and log returns for both, and compare to see if SSO daily return really doubles the daily return of SPX as claimed. Also compare the year-to-date returns to see if SSO really doubles.
 - (b) Use the following simple formulas to estimate the historical volatility $\tilde{\sigma}_k$, using M = 10, 20, 30 business days of data, for all available days indexed by k.

$$\frac{1}{M} \sum_{n=k-M+1}^{k} \frac{S_n - S_{n-1}}{S_{n-1}} = \tilde{\mu}_k \Delta t,$$
$$\frac{1}{M-1} \sum_{n=k-M+1}^{k} \left[\frac{S_n - S_{n-1}}{S_{n-1}} - \tilde{\mu}_k \Delta t \right]^2 = \tilde{\sigma}_k^2 \Delta t.$$

Plot the estimates for all available days. What can you say about the correlation between historical volatility and the SPX index itself?