Math 5760/6890
Introduction to Mathematical Finance

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What you should NOT expect to learn here:

- Predict individual stock movements
- Pick stocks to outperform the market (maybe?)
- Forecast a particular sector/market
- Predict a crash
- Anything about the future

What you hope to and will learn (if you make the efforts)

- Understand the basic market concepts such as risk, and principles such as arbitrage free trading
- Learn about different types of investment securities in terms of risks and returns
- Consolidate and extend your knowledge of time value of money (aka interest rates)
- Find financial instruments/portfolios to hedge against certain types of risks
- Price stock options - using the Black-Scholes formula, and beyond
- Learn about some important “exotic” derivatives

Lecture 1: Basic Probability Notions

- Probability triple: sample space, events and probability

- Conditional probability: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

- Independence: $P(A \cap B) = P(A) \cdot P(B)$

- Random variables, expectation: $E[X] = \sum xP(X = x)$

- Variance: $V[X] = E\left[(X - E[X])^2\right]$

- Jointly distributed RV's, covariance and correlation

$$Cov(X, Y) = E\left[(X - E[X]) \cdot (Y - E[Y])\right]$$

$$\rho(X, Y) = Corr(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X) \cdot Var(Y)}}$$

Basic Probability Notions (continued)

- Conditional expectation

$$E[X|Y] = E[X|Y = y] = \sum_x xP(X = x|Y = y)$$

- A trivial but important observation

$$E[X] = E\left[E[X|Y]\right]$$

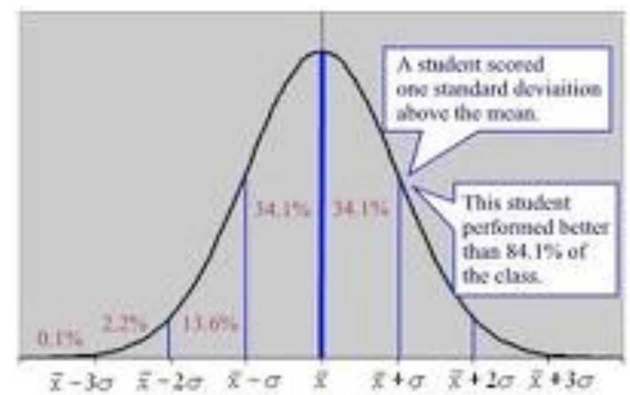
- Continuous random variable - need to consider

$$P(a \leq X \leq b)$$

Normal Random Variables

- Bell-shaped density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



- Centered at the mean
- Spread measured by the variance
- Adding the variances for two independent normal rv's
- Lognormal distribution $Y = e^X$

$$E[Y] = e^{\mu + \frac{1}{2}\sigma^2} \qquad Var(Y) = e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2}$$

Central Limit Theorem

- The most important theorem in probability theory
- Begin with any distribution (with finite mean and variance)
- A natural introduction to normal rv's
- Sum of iid (independent, identically distributed) rv's
- Properly scaled (square root of n)
- Converge in distribution

Defining Financial Risk

- Investing in an asset, what to receive in future return?
- Uncertain - could be higher, or lower than what you paid for; Investors demand **higher** *expected* return for taking **more** risk.
- How to measure risk (in financial markets)? level of uncertainty measured in terms of fluctuation - **volatility**
- Any riskless securities? Government bonds (for some governments); why would investors sometimes even pay the government to keep their money?
- Leverage - **amplify** your risk exposure. How? **Borrowed** money to invest - very risky business. Why? What do investors get out of it?

Market Efficiency, or Inefficiency

- Cornerstone of the market economy foundation
- What it says: in a **free** market, all information is already included in the **current** price of the asset (think of Markov property)
- Price reflects all past information (weak form) vs. price even reflects hidden info (strong form)
- Discovering market inefficiency is one way to explore money making opportunities, such as insider trading
- Current price = Expected future value? Need best/worst scenarios + interest
- Risk level incorporated in price: How high is the stake? Investor's appetite for risk (risk aversion, risk seeking, or risk neutral?)

Types of Investment Securities

- Equity: stocks - public and limited liability - expect higher returns
- Fixed-Income: bonds and papers.
  - Example: guaranteed $100 payback 5 yrs from now, current price = $90 - implying return around 2.1%
- Coupons, yields ... - risk in interest rates (imagine what happens to the price if the rate goes to 3%); If you sell it before maturity, you could lose money.
- Government vs corporate (may default)
- Rating and rating agencies (S&P, Moody and Fitch)

Risk Diversification

- Different but related securities to “offset” each other “approximately” - the idea behind hedging
- Create portfolios to change risk characteristics (an example: diversification) - modeling is crucial to the success
- Systematic (market) risk vs unsystematic (specific) risk
- Financial engineering: to explore the relationships among related securities, and design portfolios to appeal to investors with specific considerations - **Managing risks**
- Factor analysis: attempts to factorize risks, focusing on a few main factors
- How to achieve all these goals? One of the approaches: financial derivatives

Lecture 2: Financial Derivatives

- An effective way to manage risks is to invest in financial derivatives
- Nature of the instruments: tradable securities, on exchange or over-the-counter
- Value: derived from the future prices of other securities, or just other values that will **only be revealed** in the future; a security like this is called an **underlying**.
- Examples:
  - forward contract: **committed** to buy a share of the underlying at time T for a predetermined price K
  - call option : **right** to buy a share of the underlying at time T for a predetermined price K
  - expiration date T, strike K all specified in the contract

Payoff Examples

- In the following examples, we look at two scenarios for an underlying (a certain stock price S=$120 or $80) at a future time T:
- (i) A call option with K=$100, T=1 month, after one month
  - If S=$120, you exercise the option by buying the stock from the other party of the contract and sell it to the market, your net gain is $20;
  - If S=$80, you just let the contract expire, since you are under no obligation to buy, the net gain is 0.
- (ii) A forward contract to buy with K=$100, T=1 month, after one month
  - If S=$120, you buy the stock from the counterparty for $100 and sell it to the market for $120, the net gain is $20;
  - if S=$80, you still need to buy from the counterparty for the price of $100, knowing that the market value is only $80. The net loss is $20.

Other Examples of Derivatives on a Stock

- Put options, futures contracts
- Invest in call if you expect the stock to go up, and look for the most effective way to benefit, or invest in put if you believe the stock is going to take a hit
- Option styles - European vs American:
  - European: you can exercise only on the expiration date T
  - American: you can exercise at **any time** before the expiration date T
  - Expiration date: 3rd Friday of the month

Use of Options: Hedge, Speculate, and Arbitrage

- Hedge example: a company expecting a payment in Euro 6 months from now, worries about FX risk
    - entering a forward contract on the Euro is the obvious way to hedge;
    - if the amount is unknown, buying put contracts on the Euro is also effective.
- Speculation example: you have insider information that the stock is going to go up, you can
    - invest $1m in stocks; or $1m in call options;
    - options will yield much higher returns, but also could result in a **total** loss!
- Arbitrage example: you find inconsistency in stock and option prices - you may be able to find some opportunity to gain without taking any risk

Main Questions in Option Pricing

- Within an option contract, K (exercise or strike price) and T (expiration date) are preset
- Pricing: how much is the worth of the contract NOW (market value)?
- Factors: it depends on the current stock price, but also
- Breakthrough: Black-Scholes formula (1973)
- Crucial parameter in the formula: volatility - a measure of the average fluctuation level; but the expected stock price growth is **irrelevant**!
- How to use the option to hedge your stock positions (or vise versa)?

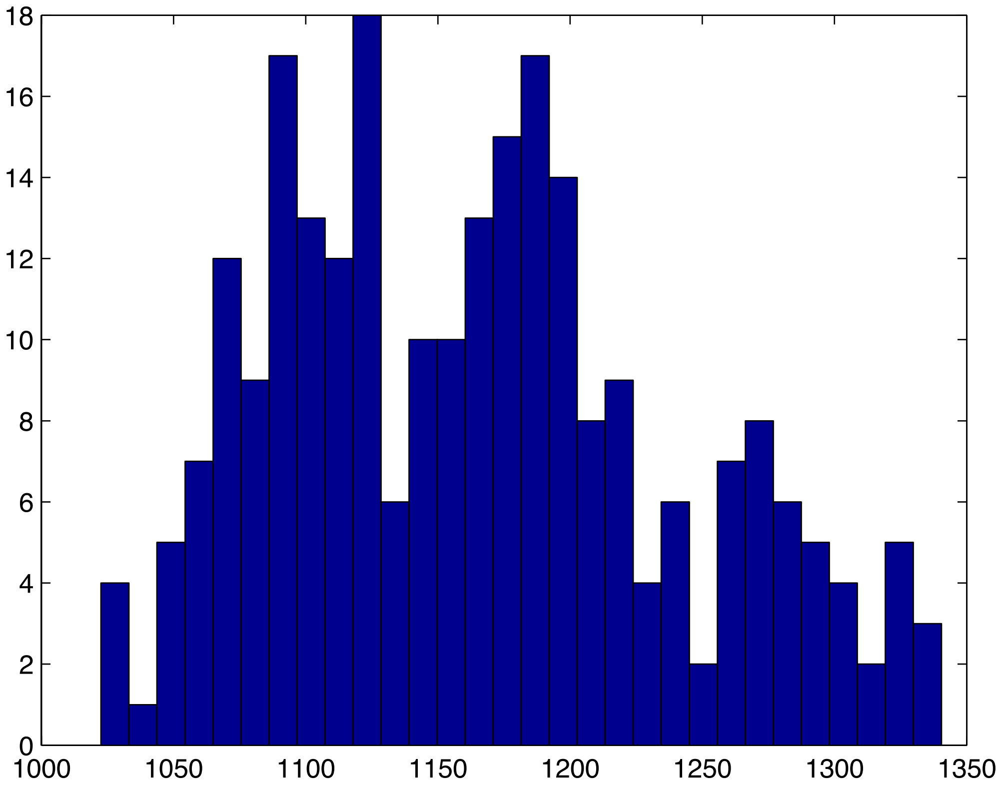
Modeling Stock Price Changes

- Focus on the **return** over a time period

$$\frac{S(t + \Delta t) - S(t)}{S(t)}$$

- Collection of returns modeled as realizations of a random variable
- Suggest a distribution for this random variable?
- Normal distribution a natural choice, or not?
- What about the stock price itself? **lognormal distribution**
- Why is normal distribution a bad choice for S itself? See the plots next

S&P 500 Price Distribution



S&P 500 Daily Return Distribution