Summary of Chapter 5

In this chapter we introduce the essential tools in Itô calculus, as we use Brownian motion to model the basic stock price evolution.

1 Brownian motion

The motivation to use Brownian motion to model security prices has its root in both financial and mathematical considerations. In financial modeling, it's the return, the relative price change, that we want to study, and in mathematical settings, it's the differential, or the increment, of the process that we want to focus. The Brownian motion can be described through the following

- random walk to begin with,
- independent increments,
- each increment is a normal random variable,

and the following are some derived properties:

- quadratic variation,
- variance of W_t proportional to t, in particular $W_t - W_s \sim N(0, t-s)$,
- $\mathrm{E}[dW_t] = 0, \quad \mathrm{Var}[dW_t] = dt$.

2 Itô's lemma

Itô's lemma tells us how to differentiate a function that depends on W_t (the function itself is smooth in its arguments). In the most intuitive form, if we have

$$dX_t = \mu\, dt + \sigma\, dW_t$$

then

$$df(X_t, t) = f_t\, dt + f_x\, dX_t + \frac{1}{2} f_{xx} (dX_t)^2 = \left(f_t + \mu f_x + \frac{1}{2}\sigma^2 f_{xx} \right) dt + \sigma f_x\, dW_t$$

3 Derivation of the Black-Scholes equation

The derivation is based on forming a portfolio, consisting of one share of the option (with price V), and α shares of S, so that the portfolio is riskless, or that the portfolio grows like a bank deposit. The last conclusion comes from the no-arbitrage principle. The equation is valid for any European derivative though

$$V_t + rSV_S + \frac{1}{2}\sigma^2 S^2 V_{SS} = rV$$

and the specific derivative is specified through its payoff function

$$V(S_T, T) = F(S_T)$$

4 Solution of the Black-Scholes equation for call/put options

The Black-Scholes formula can be obtained by solving the Black-Scholes PDE with the terminal condition

$$F(S_T) = \max(S_T - K, 0) \quad \text{for call,} \quad \text{and } F(S_T) = \max(K - S_T, 0) \quad \text{for put}$$