Summary of Chapter 3

This chapter discusses the most important and accessible model in derivative pricing: the binomial tree model.

1 A one-step, two-scenario model

This exceedingly simple and transparent model exemplifies the ideas and power of the binomial tree model. The following ideas are behind the working:

1. The model: we know what's possible (limited in a simple model) at the end of the time period, we just do not know which scenario we would end up with.

2. Securities: a), the underlying stock, b), the derivative on the underlying, and c), a cash position (zero interest rate) or a bond position (positive interest rate).

3. The use of no-arbitrage principle (in two forms): a). a riskless portfolio should earn just the riskless interest rate; b). two portfolios with matching payoffs (for each and every scenario) should have the same price.

4. Risk-neutral probabilities: those will appear after the pricing is done as the probability measure under which the current price can be expressed as an expectation.

2 Extending the one-step model to a multi-step model

There would have been no value in the model if we had not been able to extend it to multi-step models. There are two implications in the multi-step model: the increasing number of possible scenarios at the expiration time, and the even more possible price paths the model allows. The latter has more profound impact on the model: it tells the trader how a hedge can be constructed in response to the market moves.
The extension to multi-step is quite straightforward and it can be explained from two points of views.

1. backward iteration to price the derivative in questions

   A key to the pricing is the determination of the risk-neutral probabilities. The equation for the probabilities p is

$$\frac{1}{1+r\Delta t}\mathrm{E}[S_j] = S_{j-1}$$

where the stock prices on all the nodes are already generated by u and d. The backward iteration simply traces back in time to obtain

$$C_{j-1} = \frac{1}{1+r\Delta t}\mathrm{E}[C_j]$$

These expectations refer to a one-step part of the tree, namely we assume a node with stock price S_j and there are only two possible S_{j+1} prices, therefore the expectation in these equations depends on the node S_j to begin with.

2. Delta-hedge (forward in time) to find hedge ratios so the replication is achieved.

   This point of view justifies the derivative price at each node by verifying the replication. This is particularly important in daily trading practice.

3 Normal vs. lognormal models

In the normal model, each price move is described by a increase or a decrease:

$$S_{j+1} = S_j \pm \Delta,$$

while a lognormal model assumes

$$S_{j+1} = S_j e^{\pm\delta}, \quad \text{where } u = e^{\delta},\ d = e^{-\delta} = 1/u.$$

where $X_j = \pm 1$ depending on if the move is up or down. The obvious advantage of a lognormal model is that S remains positive no matter how many down steps it takes.

4 Formal expression of the binomial tree model

Assuming a lognormal model, there are two ways to describe the model formally through a formula. One focuses on the possible states at step N:

$$S_N = S_0 \exp(\delta \sum_{j=1}^{N} X_j) = S_0 \exp(Y\delta)$$

where $Y = \sum X_j$ is the difference between the number of up moves and the number of down moves, and it is a binomial random variable, from which the name of the model comes. The option price is therefore

$$C_0 = e^{-rT}\mathrm{E}\left[\max(S_0 e^{\delta Y} - K, 0)\right].$$

The other description focuses on just one step:

$$\log S_{j+1} = \log S_j \pm \delta = \log S_j + \delta X_j$$

5 Letting number of steps to approach infinity

When N goes to infinity, the option price above as an expectation is calculated with Y approaching a normal random variable. The limiting result is the celebrated Black-Scholes formula for an European call option. The put option price can be derived readily.