Summary of Chapter 1

This chapter introduces some of the most fundamental concepts in mathematical finance.

1 Risks

What is a risk? A typical perception is that a risk refers to a prospect of losing money for an investment, weather it's an investment in real estate, or financial assets. In the investment world, a sure loser is not considered a risk as no sound investor is foolish enough to engage in such an activity. The real risk is in something that we are not sure about the future. A better description is probably the word *uncertainty*. If we follow the major financial events in recent times, we note that the biggest market panics have been associated with some types of uncertain outlooks about the future. In these situations, there is no agreement about what is about to come, but usually there is some consensus about the *magnitude* of fluctuation of the market.
Given that risks are all about uncertainties, there is no better way to model them than using the central object in probability: random variables. The level of fluctuations is naturally measured by the variance of the random variable, thus the concept of volatility becomes central to the studies in mathematical finance.

2 Investment returns

The first object that we model using a random variable is the return of an investment in an asset over certain time period:

$$R = \frac{S_1 - S_0}{S_0}$$

where S_0 and S_1 are the prices at the beginning and the end of the period, respectively.
The objective for any investor is to maximize the return, with some concern over the risk. In a simplest setting, this can be formulated as a constrained optimization problem:

$$\max \mathrm{E}[R], \quad \text{subject to some constraint on } \mathrm{Var}(R)$$

3 Types of financial assets

We will discuss three types of financial assets and their pricing/hedging according to its risk behavior.

1. Risk-less assets

A bond is a security issued by a government or corporate entity that **promises** to pay certain amount (say \$1 in our settings) on a specified date in the future (maturity). As long as the interest rate is positive, the price of a bond before its maturity should always be less than the face value (\$1), and equals to the face value at maturity. In the real world a bond often pays coupons over the holding period, and for illustration purposes here we just consider those without any coupons (so-called zero-coupon bonds). In a world of fixed interest rate r (annualized), a zero-coupon bond has the price $\exp(-rT)$ where T is the time-to-maturity measured in years.

US government bonds are considered risk-free investments since the returns are always guaranteed, and they are considered in our discussions mostly for parking the cash or taking a loan in various trading schemes.

2. Risky assets

   The other type of financial assets that will be considered are obviously stocks and their derivatives. In contrast to risk-less investments like bonds, an investment in stocks can certainly lead to losses, therefore they are considered risky investments. A financial derivative on an underlying, like a stock, is also a risky asset, but the extra feature is its relationship with other quantities, such as its own underlying stock price. The quintessential example of financial derivatives is a call option on a stock: the right to purchase a share of the specified stock at time T for a prespecified price K, which will yield at time T a **payoff** in the amount $\max(S_T - K, 0)$ where S_T is the stock price at time T.

3. Use of derivatives

   In our prototype models, we will consider a trio of assets: a bond with price B_t with a fixed yield r, a stock with price S_t, and a call option on the stock with option price C_t. Our objective is to see if we can use two of them to *replicate* the other in this trio, just for a very short period of time. In another word, we want to explore the inner relations among the three assets. The benefit is obvious: we would be able to better manage portfolios consisting of those assets to handle the risk. There are two arguments we will use quite often:

   - use the stock and its derivative to form a portfolio that will no longer be impacted by the stock price change, this is called a hedge;
   - use the stock and a bond to form a portfolio that will *replicate* the derivative, that is, to synthesize a product that will do exactly the same thing as the derivative, in *all* future scenarios. If we can replicate an instrument, we are no longer concerned as which case would turn up, as we have every scenario covered.

4 Market participants

In the capital markets, liquidity is essential to the smooth running of economy. The market invented various products that will facilitate the exchanges for all participants. Financial derivatives in general fit that need and they help to create and expand the virtual economy we now live in. In the markets, there are those who manufacture those derivatives to suit the need of potential investors, and they are called the sell-side, typically represented by the large financial institutions. The other side, the so-called buy-side, represents the investors such as fund managers who either need to hedge their other positions, or just want to speculate. In between, there are those hedge funds who have their own take on the market and believe that they can find market inefficiencies and fill the gap. There is no lack of instances where huge amounts of money were made or lost.

In our studies, we often place ourselves in the shoes of the sell-side, where derivatives are constructed and sold to the buy-side. In order to cover the obligations, we would need to hedge our positions. A good trader would be able to match all his/her positions closely and always make the margins.