Chapter 7 - Practical Pricing of European Option

Pricing Methodologies

- Pricing by Computing the Expectation

$$\frac{O_t}{B_t} = \mathrm{E}_t\left[\frac{O_T}{B_T}\right]$$

- Pricing by Solving the Black-Scholes PDE for $O(S_t, t)$ with the payoff condition

$$O(S_T, T) = F(S_T)$$

- Other shortcuts
  - taking advantage of the relationship between payoff functions
  - expectation of an indicator function = probability of the event

How to calculate expectations?

- Method 1: backward iteration on a tree
- Discrete time model: $\log S_{j+1} = \log S_j + (r - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}X_j$
- Binomial tree: $X_j = \begin{cases} 1 & p = 1/2 \\ -1 & p = 1/2 \end{cases}$
- or $\log S_{j+1} = \log S_j \pm \sigma\sqrt{\Delta t}$, but $p = \frac{1}{2}\left(1 \pm \frac{r - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right)$
- Issues with binomial tree:
  - in case of variable sigma or state-dependent sigma, need to make sure to deal with a recombining tree;
  - as the number of steps increases, we may have zig-zag approximations
- Trinomial tree: $X_j = \begin{cases} \alpha & p_u \\ 0 & p_m \\ -\alpha & p_d \end{cases}$, need to verify $\mathrm{E}[X_j] = 0, \mathrm{var}[X_j] = 1$
- multiple solutions

Numerical Quadrature

- Given the pdf of S_T, $\mathrm{E}\left[F(S_T)\right] = \int F(x)g(x)\,dx$
- Approximation by quadrature: $\int_a^b f(x)\,dx \approx \sum f(x_i)\Delta x$
- Monte Carlo methods
  - Idea: $\mathrm{E}[Y] \approx \frac{1}{n}\sum_{j=1}^{n} Y_j$, Y_j ‘s sampled from the distribution
  - advantage: easy to simulate functions of Y
  - disadvantage: slow convergence - can be easily seen from CLT
  - “random numbers” are generated from random number generators (for standard normal, **randn** in Matlab, **normsinv** in Excel)

Box Muller algorithm

- If we need to simulate our own standard normal, from the uniform distribution, we can use the Box-Muller algorithm
  - sample independent U_1 and U_2 from Unif[0,1]
  - convert

$$X = \sqrt{-2\log U_1}\cos(2\pi U_2)$$
$$Y = \sqrt{-2\log U_1}\sin(2\pi U_2)$$

Convergence Estimate

$$\frac{1}{n}\sum_{j=1}^{n} X_j \to \mathrm{E}[X] + \sqrt{\frac{v}{n}}Z, \quad Z \sim N(0,1)$$

- If we let $\mu = \mathrm{E}[X]$

$$\frac{1}{n}\sum_{j=1}^{n} X_j = \mu + \frac{1}{\sqrt{n}}\left[\frac{1}{\sqrt{n}}\sum_{j=1}^{n}(X_j - \mu)\right]$$

- According to CLT, the sum within the bracket converges to Z ~ N(0,v)

- To reduce error, we could 1) increase n, 2) reduce v

Variance Reduction

- Find different ways to reduce v in sampling
  - anti-thetic sampling
  - moment matching
  - importance sampling
  - low-discrepancy numbers - quasi random numbers