

# Lecture 11: Ito Calculus

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# Continuous time models

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- We start from the model introduced in Chapter 3

$$\log S_j - \log S_{j-1} = \mu \Delta t + \sigma \sqrt{\Delta t} Z_j$$

- Sum it over j: 
$$\log S_N - \log S_0 = \sum_{j=1}^N \mu \Delta t + \sum_{j=1}^N \sigma \sqrt{\Delta t} Z_j$$

- Can we take the limit as N goes to infinity, while holding  $N \Delta t = T$  ?

- What is the benefit?

- The last sum converges to a normal random variable, so the stock price has a lognormal distribution!

- What else can we get besides the distribution of S at a fixed time?

- Increments: 
$$\log S_N - \log S_M = \sum_{j=M+1}^N \mu \Delta t + \sum_{j=M+1}^N \sigma \sqrt{\Delta t} Z_j$$

# Stock price as a process

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- Prices at different times:  $S_0, S_1, S_2, \dots, S_N$
- We must consider them as a collection (ordered) of random variables
- Obviously the order is important - when you enter at time  $j$  and exit at time  $k$ , you only care about  $\log S_j - \log S_k$ , a random variable by itself
- A collection of **time indexed** random variables is called a stochastic process
- Not only are we concerned about individual  $S_j$  as a random variable, but we also need to consider all possible increments  $\log S_j - \log S_k$  for all  $j > k$
- As for the collection of random variables, we are concerned about their distributions, but we are even more interested in the relations between different increments
- Natural assumption as a first step: all increments over nonoverlapping time periods are independent. Is this a reasonable assumption for stock prices?

# Increments

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- Stock price change over a time period
- What we get from our discrete model: a sum of independent Bernoulli rv's - binomial rv
- If we divide the time period into subintervals, we are still dealing with binomial rv's
- As the time partition increases, these binomial rv's converge to normal rv's (in distribution), justified by CLT.
- Statistics: the mean and the variance (of increments) should depend on the time elapsed:  $\mu(t_j - t_k)$  and  $\sigma^2(t_j - t_k)$
- Independent increments: as long as individual rv's are independent!

# Random walk and Markov property

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- Use notation  $X_j = \log S_j$
- A sum of steps, each consisting of two components (drift + Z)
- Called a random walk,  $X_j$  is the position of the walk at time j
- Increments  $X_j - X_k$ , independent of all the previous X's before k
- Distribution of X at j, given X at k, is unaffected by the X values before k
- Dependence of the history up to k - only through X at k
- This is called the Markov property!

# From random walk to Brownian motion

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- Think of the limiting process as  $N \rightarrow \infty, \Delta t \rightarrow 0, N\Delta t = T$
- $X_j = X_{t_j} \rightarrow X_t$ , collection of rv's indexed by a continuous time variable  $t$
- Properties inherited or extended:
  - $X$  at  $t$  is a normal random variable;
  - increment  $X_t - X_s$  is normal random variable  $N(\mu(t-s), \sigma^2(t-s))$
  - nonoverlapping increments are independent
  - The path,  $X$  as a function of  $t$ , is continuous, but nowhere differentiable
- Standard notation:  $W_t$

# Definition of BM

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- A process  $W_t$  indexed by  $t$  for  $t \geq 0$  is a Brownian motion if  $W_0 = 0$ , and for every  $t$  and  $s$  ( $s < t$ ), we have  $W_t - W_s$  is distributed as a normal random variable with variance  $t-s$ , and the distribution of  $W_t - W_s$  is independent of the behavior of  $W$  before  $s$ .
- Notice the above says much more than that  $X_t = \sqrt{t}Y$  where  $Y = N(0, 1)$
- Quadratic variation
  - why is it that the variance is proportional to the time elapsed?
  - why is that BWs are so ragged?
  - how does the stock price variance grow in time?

# Extending BMs

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- Add a (time-dependent) drift
- Allow local variance (for each step) to be time-dependent
- Discrete time:  $X_j - X_{j-1} = \mu_j \Delta t + \sigma_j \sqrt{\Delta t} Z_j$
- Continuous time:  $dX_t = \mu(t) dt + \sigma(t) dW_t$
- Stock return over (t,t+dt):

$$\frac{dS_t}{S_t} = \mu(t) dt + \sigma(t) dW_t$$

- Black-Scholes model for stock price
- Do we have  $\frac{dS_t}{S_t} = d \log S_t$  ?



# Ito's lemma

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- $f(x)$  is a continuously twice differentiable
- usual differential:  $df = f'(x) dx$
- if  $x=x(t)$  is also continuously differentiable:  $df = f'(x) x'(t) dt$
- now let  $x=X_t$  as described on the previous slide
- notice  $W_t$  is nowhere differentiable
- guess:  $df(X_t) = f'(X_t) dX_t = f'(X_t) (\mu dt + \sigma dW_t) \quad ?$
- not quite! as we see
$$W_{t+h}^2 - W_t^2 = (W_{t+h} - W_t)(W_{t+h} + W_t)$$
$$= 2W_t(W_{t+h} - W_t) + (W_{t+h} - W_t)^2$$
- expect  $dW_t^2 = 2W_t dW_t + dt$

# From Taylor expansion

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- Assuming  $f(x)$  twice differentiable

$$f(X_{t+h}) = f(X_t) + f'(X_t)(X_{t+h} - X_t) + \frac{1}{2}f''(X_t)(X_{t+h} - X_t)^2 + \dots$$

- Ito process:  $dX_t = \mu dt + \sigma dW_t$

$$X_{t+h} - X_t = \mu h + \sigma \sqrt{h}Z + e$$

$$(X_{t+h} - X_t)^2 = \mu^2 h^2 + \sigma^2 h Z^2 + 2\mu\sigma h^{3/2}Z + \dots$$

- Leading term:  $\sigma^2 h$
- Justifications: the difference has mean and variance:

$$\sigma^2 h E[Z^2 - 1] = 0, \quad \sigma^4 h^2 \text{Var}(Z^2 - 1) = 3\sigma^4 h^2$$

# Ito's lemma

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- Letting  $h \rightarrow dt$
- Assuming differentiability again

$$d(f(X_t)) = \left( f'(X_t)\mu + \frac{1}{2}f''(X_t)\sigma^2 \right) dt + \sigma f'(X_t) dW_t$$

- If we allow  $f$  to be time dependent

$$d(f(X_t, t)) = \left( f_t(X_t, t) + f_x(X_t, t)\mu + \frac{1}{2}f_{xx}(X_t, t)\sigma^2 \right) dt + \sigma f_x(X_t, t) dW_t$$

- Theorem 5.1 (page 110) notations

$$dt^2 = 0$$

$$dt dW_t = 0$$

$$(dW_t)^2 = dt$$

# Applications

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- Product rule: let  $X_t$  and  $Y_t$  to be Ito processes

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t dY_t$$

- If  $dX_t = \mu_1 dt + \sigma_1 dW_t$

$$dY_t = \mu_2 dt + \sigma_2 dW_t$$

- then  $dX_t dY_t = \sigma_1 \sigma_2 (dW_t)^2 = \sigma_1 \sigma_2 dt$

- What about

$$d\left(\frac{X_t}{Y_t}\right)$$

# Applications in stock price modeling

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- Solving SDE  $\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$

- Try  $f(S_t) = \log S_t$

$$\begin{aligned} df(S_t) &= \frac{1}{S_t} dS_t + \frac{1}{2} \left( -\frac{1}{S_t^2} \right) \sigma^2 S_t^2 dt \\ &= \left( \mu - \frac{1}{2} \sigma^2 \right) dt + \sigma dW_t \end{aligned}$$

- Integrate in t, assuming constant mu and sigma

$$\log S_T - \log S_0 = \left( \mu - \frac{1}{2} \sigma^2 \right) T + \sigma W_T$$

$$S_T = S_0 \exp \left[ \left( \mu - \frac{1}{2} \sigma^2 \right) T + \sigma W_T \right]$$

# CEV model

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- Assuming volatility is S dependent

$$\frac{dS_t}{S_t} = \mu dt + S_t^{\beta-1} \sigma dW_t$$

- $0 < \beta < 1$  implies that the volatility is inverse proportional to S

- $f(S) = \frac{S^{1-\beta}}{1-\beta}$ , Ito's lemma gives

$$d(f(S_t)) = \left( S^{1-\beta} \mu - \frac{\beta}{2} S^{\beta-1} \sigma^2 \right) dt + \sigma dW_t$$

- No luck in explicit solution unless beta=1

# Deriving Black-Scholes Equation

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- Consider the pricing of a call option, with strike  $K$ , expiration  $T$
- Assume  $S$  follows a geometric BM
- Risk free interest rate  $r$
- At time  $t < T$ , the price of call is a function of stock price at the time ( $S$ )
- Recognizing  $C = C(S, t)$

$$\begin{aligned} dC(S_t, t) &= \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S} dS_t + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} (dS_t)^2 \\ &= \left( C_t + \mu S C_S + \frac{1}{2} \sigma^2 S^2 C_{SS} \right) dt + \sigma S C_S dW_t \end{aligned}$$

# Deriving Black-Scholes Equation (continued)

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- Forming a portfolio: one share of call + alpha shares of the stock
- Change of the portfolio over (t,t+dt), assuming constant alpha:

$$d(C + \alpha S) = \left( C_t + \mu S C_S + \frac{1}{2} \sigma^2 S^2 C_{SS} + \alpha \mu S \right) dt + \sigma S (C_S + \alpha) dW_t$$

- If we choose  $\alpha = -C_S$  (delta hedging), the random component disappears, which implies that the portfolio is hedged - no effect of stock price fluctuation
- Risk-free? We must have  $d(C + \alpha S) = r(C + \alpha S)dt$
- This leads to the Black-Scholes PDE and terminal condition

$$C_t + r S C_S + \frac{1}{2} \sigma^2 S^2 C_{SS} = r C \qquad C(S_T, T) = \max(S_T - K, 0)$$



# Use of the PDE

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- The PDE is parabolic, solutions are smoothed
- Set up a region in  $(S,t)$ :  $0 < t < T$ ,  $0 < S < S_{\max}$
- Terminal condition imposed at  $t=T$
- Solve backward in time from  $T$  to  $0$ :  $C(S,0)$
- Enter the observed current price  $S(0)$  for  $S$
- Boundary conditions:  $C(0,t) = 0$ ,  $C(S_{\max},t) = S_{\max} - K$
- Advantages of the PDE approach:
  - easy to extend to time-dependent parameters
  - efficient numerical methods available

# Justification of the derivation

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- How do we justify this price (solution from a PDE)?
- Imagine you start with  $C(S,0)$ , when the stock price is  $S$ , by following the delta hedge strategy, you want to end up with  $\max(S_T - K, 0)$ , no matter what happens to the market
- Replication strategy: invest  $C(S,0)$  in stock and the risk-less bond, adjusting according to the call delta, verify at  $T$  that the total portfolio value matches the payoff of the call option
- Composition of the portfolio: alpha shares of the stock, beta units of the bond

$$P(t) = \alpha(t)S(t) + \beta(t)B(t)$$

- $\alpha(t)$ ,  $\beta(t)$  to be adjusted, according to the strategy

# Change of value in the portfolio

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- Change of portfolio value in time:  $P(t + \Delta t) - P(t)$
- In differential:  $dP = \alpha(t)dS(t) + \beta(t)dB(t) + S(t)d\alpha(t) + B(t)d\beta(t)$

- In discrete form:

$$\begin{aligned} & \alpha(t + \Delta t)S(t + \Delta t) - \alpha(t)S(t) \\ &= \alpha(t + \Delta t)S(t + \Delta t) - \alpha(t)S(t + \Delta t) + \alpha(t)S(t + \Delta t) - \alpha(t)S(t) \\ &= (\alpha(t + \Delta t) - \alpha(t))S(t + \Delta t) + \alpha(t)(S(t + \Delta t) - S(t)) \end{aligned}$$

$$\begin{aligned} & \beta(t + \Delta t)B(t + \Delta t) - \beta(t)B(t) \\ &= \beta(t + \Delta t)B(t + \Delta t) - \beta(t)B(t + \Delta t) + \beta(t)B(t + \Delta t) - \beta(t)B(t) \\ &= (\beta(t + \Delta t) - \beta(t))B(t + \Delta t) + \beta(t)(B(t + \Delta t) - B(t)) \end{aligned}$$

- Total change in two parts:

$$\alpha(t)(S(t + \Delta t) - S(t)) + \beta(t)(B(t + \Delta t) - B(t)) \longrightarrow \alpha dS + \beta dB$$

$$\begin{aligned} & (\alpha(t + \Delta t) - \alpha(t))S(t + \Delta t) + (\beta(t + \Delta t) - \beta(t))B(t + \Delta t) \\ & \longrightarrow Sd\alpha + Bd\beta + d\alpha dS + d\beta dB \end{aligned}$$

# Self-financing strategy

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- First part: changes in stock price, bond price, holding number of shares fixed over the time period
- Second part: adjusting the number of shares, all at the end of the time period
- Self-financing strategy: making sure the second part is zero
- This corresponds to rebalancing such that no money is taken out of the portfolio, no money is injected into the portfolio
- Such is the name of the strategy: self-financing
- Consequence of this trading strategy:

$$dP = \alpha dS + \beta dB$$

# Replicating the call

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- Begin with a portfolio  $P = \alpha(0)S(0) + \beta(0)B(0) = C(S(0), 0)$
- Following  $\alpha = \frac{\partial C}{\partial S}$ , and a beta such that it is a self-financing strategy
- Want to show  $P(T) = C(S(T), T)$ , no matter what  $S(T)$  it ends up with
- Consider the differential

$$\begin{aligned}d(P(S, t) - C(S, t)) &= dP - dC \\&= \frac{\partial C}{\partial S}dS + \beta dB - \frac{\partial C}{\partial t}dt - \frac{\partial C}{\partial S}dS - \frac{1}{2} \frac{\partial^2 C}{\partial S^2} (dS)^2 \\&= \beta r B dt - \frac{\partial C}{\partial t} dt - \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \sigma^2 S^2 dt\end{aligned}$$

- We use  $\frac{dS}{S} = \mu dt + \sigma dW$ ,  $dB = rBdt$ ,  $P = \frac{\partial C}{\partial S}S + \beta B$ , and the BS equation
- Result:  $d(P - C) = r(P - SC_S)dt - r(C - SC_S)dt = r(P - C)dt$

# Matching at T

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- Solving the ODE:  $P(t) - C(t) = (P(0) - C(0)) e^{rt} = 0$
- We have  $P(S, t) = C(S, t)$ , for  $0 < t \leq T$ , **the call is replicated!**
- Need to check the self-financing condition
- Theorem 5.3:
  - A unique beta exists, given alpha is a smooth function of S and an initial portfolio value  $P(0)$ , such that  $P = \alpha S + \beta B$  is a self-financing portfolio with initial value  $P(0)$ .
- Implication on the hedging practice: by the end of the trading adjustment period, the rebalancing needs to observe the condition: there can be only transfer of money in-between the stock and bond accounts

# Solving the PDE

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- Linear PDE, variable coefficients
- A series of changes of variables to simplify the equation
- First,  $S = e^Z$ , we arrive at a constant-coefficient equation

$$\frac{\partial C}{\partial t} + \left(r - \frac{1}{2}\sigma^2\right) \frac{\partial C}{\partial Z} + \frac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial Z^2} = rC$$

- Change of time variable  $\tau = T - t$

$$\frac{\partial C}{\partial \tau} - \left(r - \frac{1}{2}\sigma^2\right) \frac{\partial C}{\partial Z} - \frac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial Z^2} = rC$$

- $C = e^{-r\tau} D$

$$\frac{\partial D}{\partial \tau} - \left(r - \frac{1}{2}\sigma^2\right) \frac{\partial D}{\partial Z} - \frac{1}{2}\sigma^2 \frac{\partial^2 D}{\partial Z^2} = 0$$

# Heat Equation

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- To eliminate the first derivative term

$$y = Z + \left( r - \frac{1}{2}\sigma^2 \right) \tau$$

- Standard heat equation

$$\frac{\partial D}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 D}{\partial y^2}$$

- Initial condition is also likewise transformed
- Solution transformed into the original variables
- Black-Scholes formula reproduced



# Dividend-paying stock

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- The previous model assumes no dividend paying stocks
- Many stocks do pay dividend
- FX products - foreign currency as the underlying and it grows at its risk-free rate
- The following argument assumes reinvestment - dividends back into shares
- If the dividend rate is  $d$ , one share at  $t$  will grow to  $\exp(d(T-t))$  shares at  $T$
- Buying  $\exp(-d(T-t))$  shares at  $t$  is equivalent to one futures contract:
- Price of futures contract:  $S(t)e^{-d(T-t)} - Ke^{-r(T-t)}$
- $X(t) = S(t)e^{-d(T-t)}$  can be viewed as the delivery contract price

# Call option on X (dividend paying)

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- An option on X with expiration T must have the same value as an option on S
- But the delivery contract pays no dividend (X is its price)
- Process for X:  $\frac{dX_t}{X_t} = (\mu + d) dt + \sigma dW_t$
- Drift does not matter!
- Call price: 
$$C(S, t) = X N(d_1) - K e^{-r(T-t)} N(d_2)$$
$$= S e^{-d(T-t)} N(d_1) - K e^{-r(T-t)} N(d_2)$$
- with  $d_1 = \frac{\log(S/K) + (r - d + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}$
- Applies to commodity options - it costs money to hold commodities ( $d=-q$ )