Lecture 9: Practicalities in Using Black-Scholes

Major Complaints

- Most stocks and FX products don’t have log-normal distribution
- Typically fat-tailed distributions are observed
- Constant volatility assumed, while implied volatility as observed from the market is clearly stochastic
- Volatility skew/smile evident
- Need a volatility surface to accurately price options (in K-T plane)
- Dynamic hedging could be expansive

Trading volatility

- Option values move strongly dependent on the underlying (stock) movements
- What ***vol*** to put in the formula when pricing options?
- For market participants, one extra variable “vol” to bet on
- Historical vol - how is it estimated? sample size vs. relevance
- Implied vol - the sigma value used in B-S formula to yield the option price that matches the market price
- Investment principle: same “buy low, sell high”, except it’s on the vol

Implied volatility

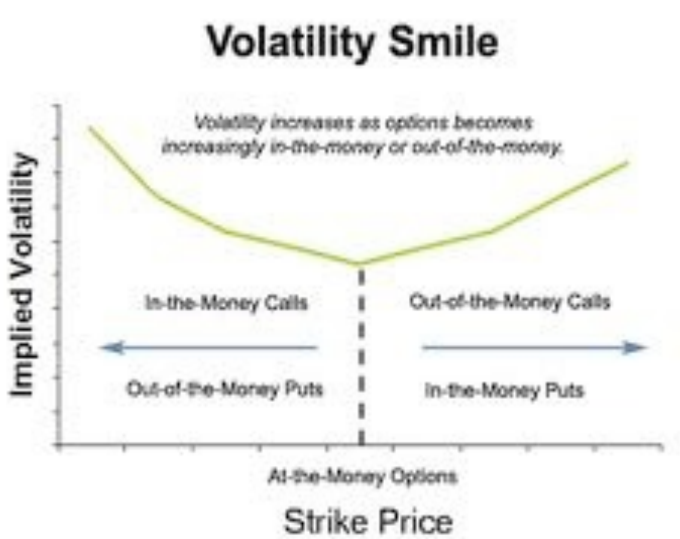
- Suppose there is a call traded actively on the market, with a quoted price
- Option parameters: strike K, maturity T, interest rate r -- all well observed
- Except the vol, which has a huge impact on the price, but not observable
- Implied vol is the value of sigma σ_{imp} such that

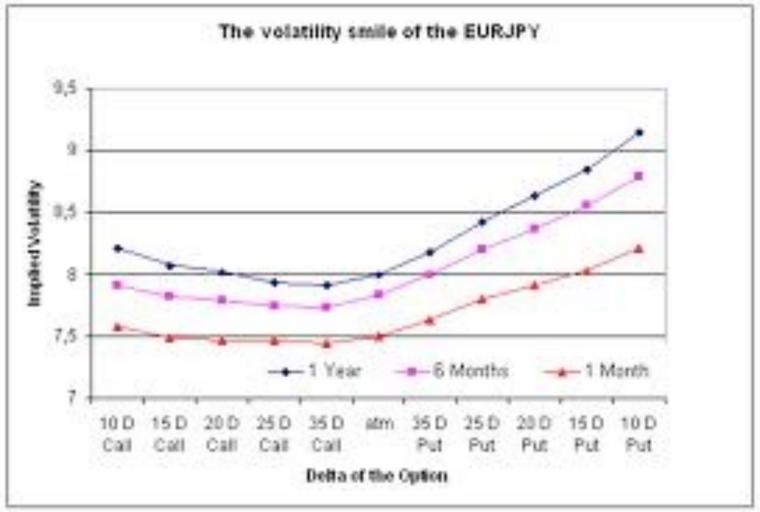
$$C_{BS}\left(S, K, T, r; \sigma_{imp}\right) = C_{market}$$

- Well defined through, B-S formula, as C_{BS} is monotone in σ
- If the B-S model had been realistic, we should have observed the same sigma in all options on the same underlying

Volatility Smile

- But we don't! σ_{imp} for different strikes, different maturities turned out to be different, even for the same underlying
- Explanations:
  - Supply and demand
  - Out-of-the-money options may be more valuable than Black-Scholes formula indicates - higher probabilities - fat tails
  - Stock price - volatility correlations, typically negative





The Greeks

- Consider C_BS as a function of several variables
- Partial derivatives of the option price with respect to individual variables can be interpreted with financial interpretations - they are called the “Greeks”
- As situations change, the variables/parameters in the formula will change - leading to price changes
- Assume **small** variable/parameter changes, 1st order Taylor expansion may be sufficient to capture the majority of option price change

$$F(S + \delta S, t + \delta t, \sigma + \delta \sigma) = F(S, t, \sigma) + \delta S \frac{\partial F}{\partial S} + \delta t \frac{\partial F}{\partial t} + \delta \sigma \frac{\partial F}{\partial \sigma} + h.o.t.$$

- Higher order approximation may be needed!

The Greeks of a call

- Delta

$$\Delta = \frac{\partial C}{\partial S} = N(d_1)$$

- Gamma

$$\Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{N'(d_1)}{S\sigma\sqrt{T-t}}$$

- Vega

$$\mathcal{V} = \frac{\partial C}{\partial \sigma} = S\sqrt{T-t}N'(d_1)$$

- Theta

$$\Theta = \frac{\partial C}{\partial t} = -\frac{SN'(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-rT}N(d_2)$$

- Rho

$$rho = \frac{\partial C}{\partial r} = KTe^{-rT}N(d_2)$$

Use of Gamma

- Include the second-order effects:

$$C(S + \Delta S, K, t, \sigma) = C(S, K, t, \sigma) + \Delta \cdot \Delta S + \frac{1}{2}\Gamma(\Delta S)^2 + ...$$

- How do we make the second order correction?
- Set up the portfolio so that not only the delta is zero, but also the gamma
- Notice that the gammas for calls and puts are both positive
- It can be used to prove that the BS price of a call/put is a convex function of the spot stock price S
- Note "spot" here means the current observed underlying (stock) price

Beyond Black-Scholes

- Need models to address
  - non-lognormal distribution - fat tails
  - vol skew/smile
  - stochastic volatility
  - jumps

Other Models

- Jump models:
  - jump time - modeled by Poisson random variables
  - jump size - either fixed, or modeled by a random variable (normal, or double exponential, etc.)
  - random jump size: impossible to hedge
- Jump diffusion model: price moves consisting of two components - small moves (log-normal distribution) + jumps
- Risk-neutral probability measure: non-unique price
- Practical use: jumps from actual probability, incorporated with other components to achieve risk-neutral property

Other Models (continued)

- Spot stock price dependent volatility: $\sigma = \sigma(S)$
- Time dependent volatility: $\sigma = \sigma(t)$
- Smile not explained by the above two fixes!
- Stochastic volatility is needed (both from observation and from implied vol):
    - more convenient to work with in continuous time
    - random component modeled by another Brownian motion, correlated with the random component in stock price model
    - usually exhibit mean reversion

Other Models (continued)

- Random time:
  - address the issues with calendar time vs. business time
  - not all days are created equal!
  - VG model: variance gamma - another stochastic process
  - especially useful in exotic option pricing
- Incomplete market models
  - need to study utility functions/investor behavior
  - prices are more subjective