Lectures 4-6: Interest Rates and PV Analysis

Time Value of Money and Interest Rates

- Prefer to receive payments sooner, rather than later
- Simple compounding: $\$1 \to \$(1+r)$ after one time period
- This rate is applied to a time period, and quoted according to the length of the time period (daily, monthly, yearly, etc.)
- Annualized rate: $1 + r\Delta t$, where r is annualized and Δt is measured in years
- Compounding: $\left(1+\frac{r}{n}\right)^{nt} \to e^{rt}$

Present Value Analysis

- A dollar paid in the future viewed today: should be discounted
- Stream of cash flows (payments)
- Each payment discounted by a factor $(1+r)^{-i}$ where i is the number of time periods leading to the payment date, or e^{-rt} in continuous compounding where t is the time to payment date (usually measured in years)
- PV of the stream of c.f. is therefore the sum of discounted cash flows
- In the real world the rate r is (1) different for different maturity i (or t), and (2) is time varying in some random fashion.

PV Analysis

- Cash flow stream: $\mathbf{a} = (a_1, a_2, \ldots, a_n)$
- PV of cf stream: $PV(\mathbf{a}) = \frac{a_1}{1+r} + \frac{a_2}{(1+r)^2} + \frac{a_3}{(1+r)^3} + \cdots + \frac{a_n}{(1+r)^n}$
- Can be used to compare two different cash flow streams
- Other ways to compare: Proposition 4.2.1
- The crucial role played by the interest rate: one stream is favored over another based on the interest rate used.

Interest rate examples

- Making deposit for the next 20 years, expect retirement payments of $1000 per month for the 30 years afterwards. How much should we save now? Note what a nonsense this question is since the future rates are anything but known to us at this time.

- Mortgage payment calculation: this is for real as the rate quoted in the mortgage contract is fixed, and we are here to find the monthly payment amount.

- A decomposition of the monthly payment: interest + principal, the interest portion covers the interest on the remaining principal incurred for one month.

- Interest portion high at beginning, reduced to zero at the end.

- Principal reduction initially low, becoming more dominant towards the end.

Rate of return

- Invest a, receive b: $a \rightarrow b : r = \frac{b}{a} - 1$
- Quoted in percentage
- Generalized to a stream of returns: r solves the equation P(r)=0, where P is

$$P(r) = -a + \sum_{i=1}^{n} b_i (1+r)^{-i}$$

- Making sure unique solution exists
- Guaranteed by all positive returns
- Numerical tools needed to solve for the nonlinear equation

Continuously Varying Interest Rates

- Defining short rate r(s): applying to period (s,s+h) where h is very small

$$\$1 \to \$(1 + r(s)h)$$

- Announced at time s, also called spot or instantaneous interest rate.
- Introduce D(t): the amount you will have on account at time t if you deposit 1 at time 0.
- D(t) grows in time if r is positive.
- For small h, using differentials to describe is convenient.

Deriving Equations for D(t)

- For small h

$$D(s+h) - D(s) \approx D(s)r(s)h$$

- Let h go to zero

$$D'(s) = D(s)r(s)$$

- Solution

$$D(t) = \exp\left\{\int_0^t r(s)\,ds\right\}$$

- Bond price

$$P(t) = \frac{1}{D(t)} = \exp\left\{-\int_0^t r(s)\,ds\right\}$$

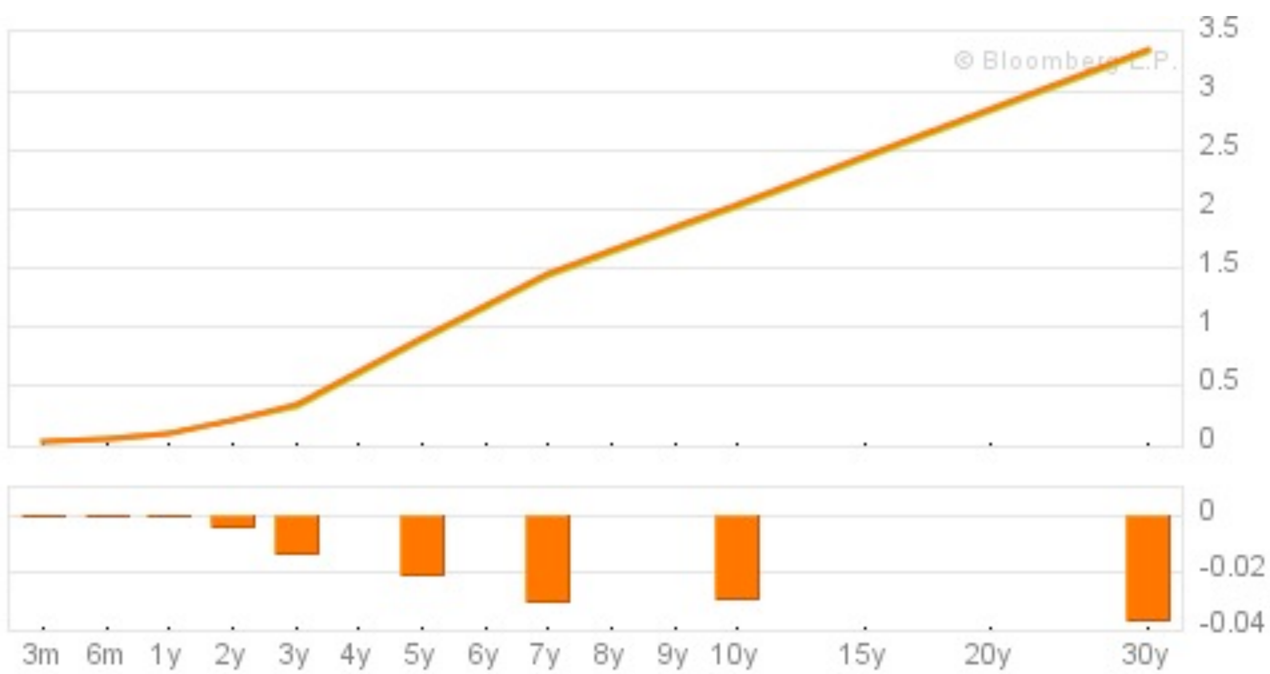
- Face the reality: r(t) is random so stochastic models are needed and bond prices will be given in terms of expectations.

Yield Curve

- Average of the spot interest rate up to t

$$\bar{r}(t) = \frac{1}{t}\int_0^t r(s)\,ds$$

- This function of t plotted is called the yield curve.



How Is Yield Curve Used

- One dollar paid in t years, PV is $e^{-\bar{r}(t)t}$
- Bond price: how much are you willing to pay for a dollar paid (guaranteed) by the issuer (government or corporate) in t years: P(t)
- Yield of the bond price: $-\frac{1}{t}\log P(t)$ (assuming no coupons)
- Yield curve constructed from several bond prices, then smoothly connected
- Shape of the yield curve
- Yield curve changes all the time

Random Interest Rates

- Short rate r(s) revealed at time s only
- Stochastic modeling of r(s) needed
- Interest rate models for r(s) start with applications of Brownian motion
- Bond price

$$P(t) = E\left[\exp\left\{-\int_0^t r(s)\,ds\right\}\right]$$