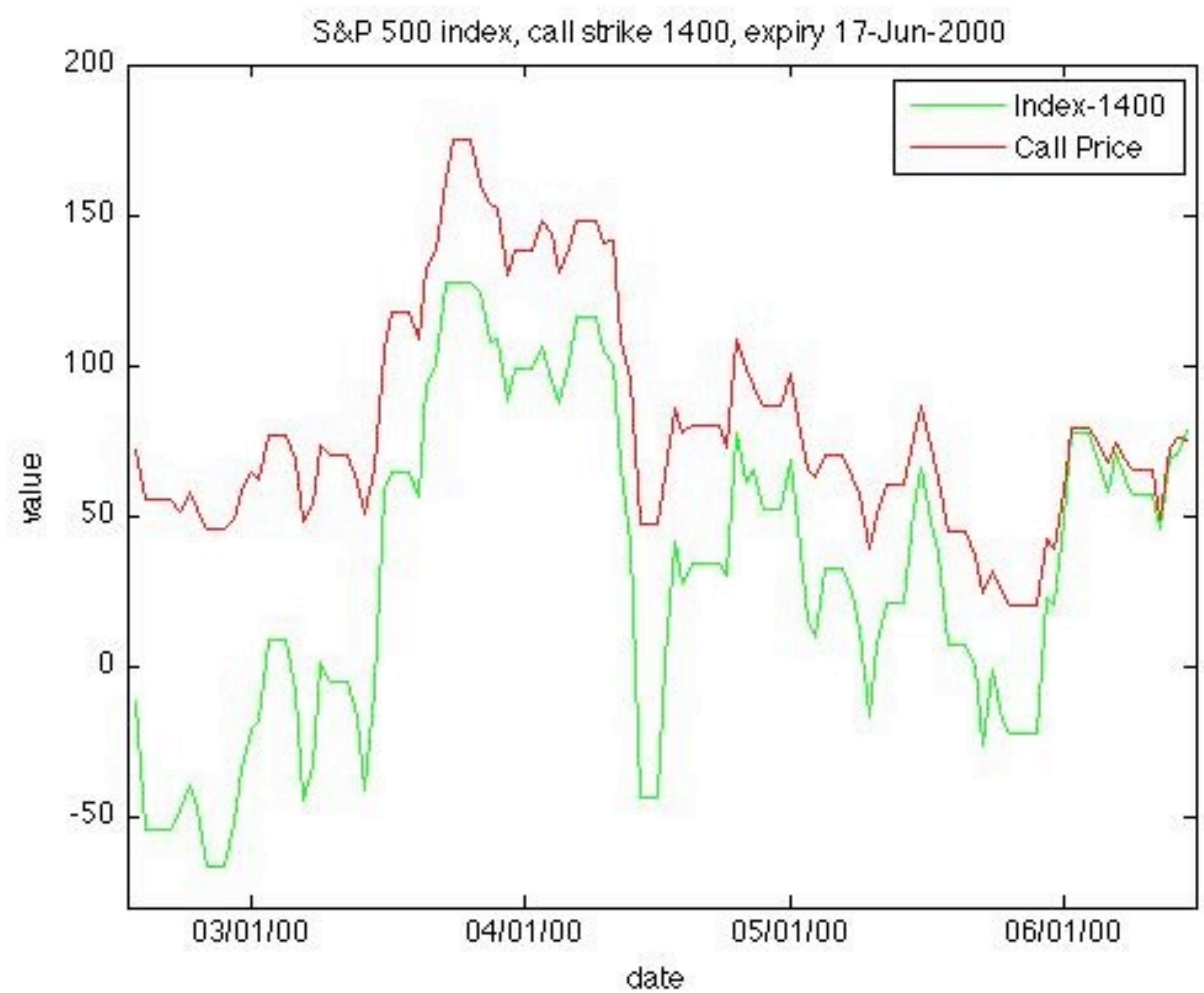
Lectures 2 and 3

Examples of call prices in relation to underlying



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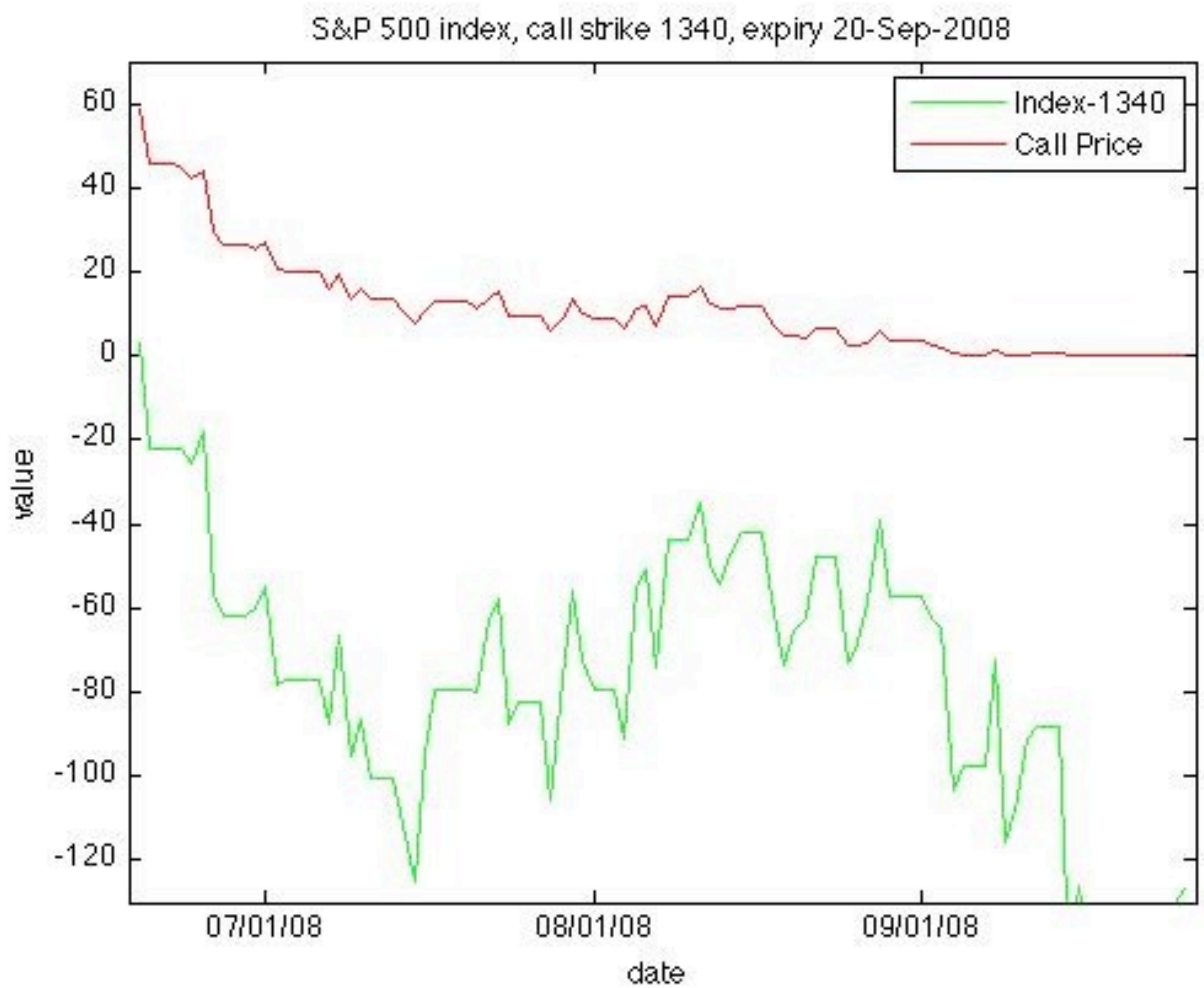
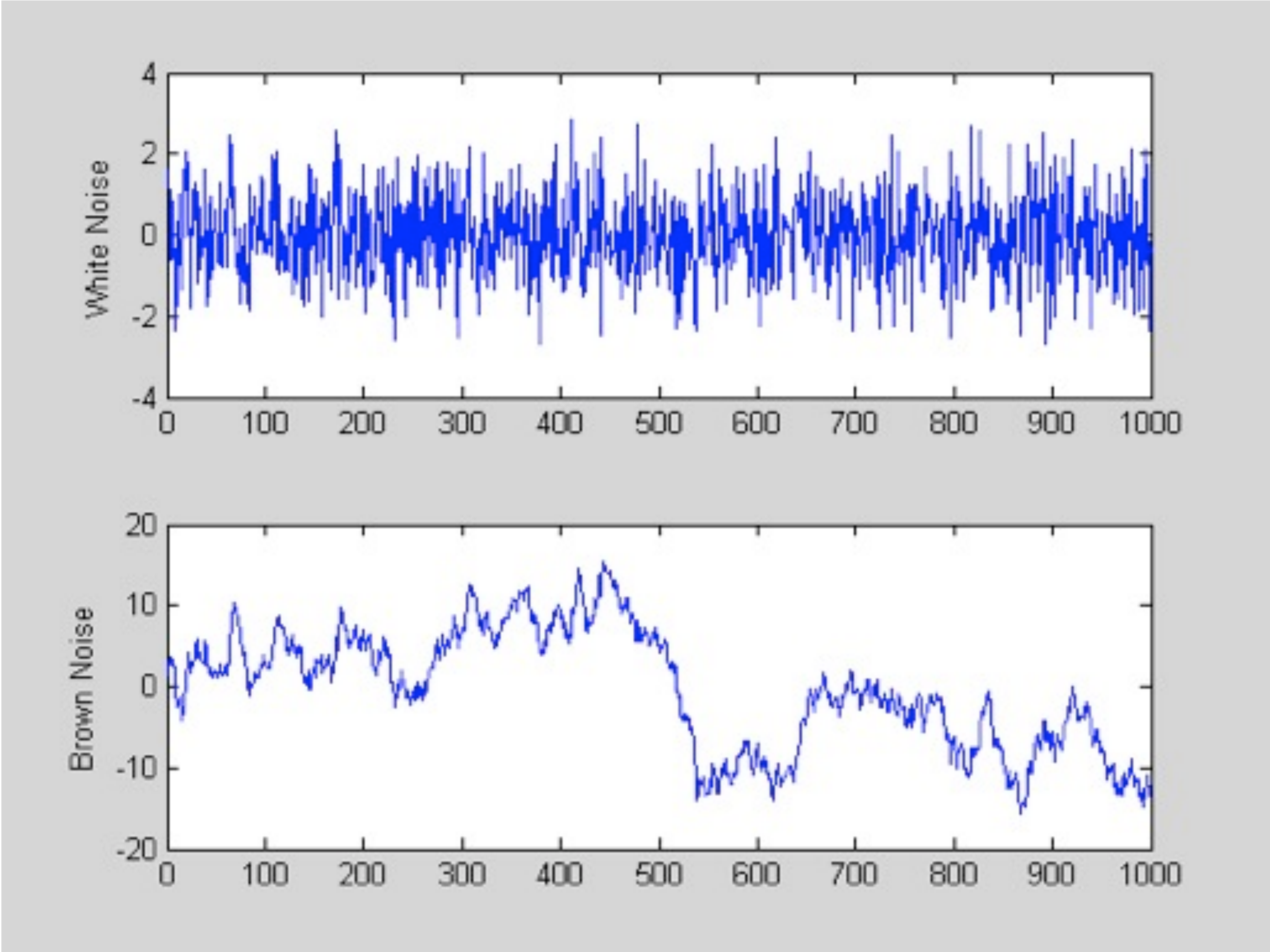


Brownian Motion

- $X(t), t \geq 0$: a collection of rv's indexed by t;



- White noise: the increments

Introducing Brownian Motion

- Focusing on **returns** over any period of time

$$R_i = \frac{S_{i+1} - S_i}{S_i}$$

- Reasonable to assume R_i to have normal distribution

$$R_i \sim N(\mu \Delta t, \beta^2) \quad \text{or} \quad R_i = \mu \Delta t + \epsilon_i$$

- What can we say about ϵ_i ?

$$\epsilon_i \sim \beta N(0, 1)$$

Brownian Motion Definition

- $X(t), t \geq 0$ is said to be a Brownian motion with drift μ and variance σ^2 if
    - $X(0)$ is a given constant
    - For all positive y and t, $X(t+y) - X(y)$ (the increment) is independent of the process up to time y and has a normal distribution with mean μt and variance $\sigma^2 t$.
- Implications:
    - move step by step, each step a normal rv independent of previous steps
    - each step with mean and variance proportional to t.

Important properties

- X(t) is continuous:

$$\lim_{h \to 0} (X(t+h) - X(t)) = 0$$

- Nowhere differentiable

- To see it as a limiting process of a random walk, we introduce

$$X_i = \begin{cases} 1, & \text{head} \\ -1, & \text{tail} \end{cases}$$

- with sum till n: $Y_n = X_1 + X_2 + \cdots X_n$

- This is called a symmetric random walk if head and tail have the same probability 1/2

Convergence

- As Δ gets smaller and smaller,
- $X(t) - X(0)$ converges to a normal random variable
- mean = μt , variance = $\sigma^2 t$
- the collection of process values over time becomes a Brownian motion process with drift μ and variance parameter σ^2
- Theorem 3.2.1: Given X(t)=x, the conditional probability law of the collection of prices $X(y), 0 \leq y \leq t$, is the same for all values of μ .

Random Walk

- let Δ to be a small time increment
- head probability $p = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta}\right)$ and tail probability $1 - p$
- Consider the process

$$X(n\Delta) = X(0) + \sigma\sqrt{\Delta}\,(X_1 + X_2 + \cdots X_n)$$

- or for $n = t/\Delta$

$$X(t) - X(0) = \sigma\sqrt{\Delta}\sum_{i=1}^{t/\Delta} X_i$$

- We can compute

$$E[X(t) - X(0)] \qquad Var[X(t) - X(0)]$$

Geometric Brownian Motion

- Problem with Brownian motion modeling stock prices:
  - negative values possible
  - price difference over an interval has the same normal distribution no matter what the price at the beginning of the interval
- To fix these two problems, consider $S(t) = e^{X(t)}$ where $X(t)$ is a Brownian motion with drift μ and variance parameter σ^2
- $S(t)$ is said to be a geometric Brownian motion process
- increment in log: $\log\left(\frac{S(t+y)}{S(y)}\right) \approx R(y)$

Useful results with geometric Brownian motion

- σ is called the volatility parameter
- If $S(0) = s$, we can write $S(t) = se^{X(t)}$
- If X is normal,

$$E[e^X] = \exp\left(E[X] + Var[X]/2\right)$$

$$E[S(t)] = se^{\mu t + \frac{1}{2}\sigma^2 t}$$

- In terms of random walk: $S(y+\Delta) = S(y)e^{X(y+\Delta)-X(y)}$, with move factors

$$u = e^{\sigma\Delta}, \ d = e^{-\sigma\Delta}$$

The Maximum Variable

- Let $X(v), v \geq 0,$ be a Brownian motion
- Consider the maximum process $M(t) = \max_{0 \leq v \leq t} X(v)$
- What we know about this rv for a fixed t?
- Theorem 3.4.1 (conditional distribution):

$$P(M(t) \geq y | X(t) = x) = e^{-2y(y-x)/t\sigma^2},\ y \geq 0$$

- Corollary 3.4.1 (unconditional distribution):

$$P(M(t) \geq y) = e^{-2y\mu/\sigma^2} \bar{\Phi}\left(\frac{\mu t + y}{\sigma\sqrt{t}}\right) + \bar{\Phi}\left(\frac{y - \mu t}{\sigma\sqrt{t}}\right)$$

The Cameron-Martin Theorem (Girsanov)

- Expectation under different measures
- Reflected in different drifts $\mu \neq 0, \ \mu = 0$
- Sometimes one expectation $(\mu = 0)$ is much easier to obtain than the other
- Theorem 3.5.1: Let W be a rv whose value is determined by X (BM) up to t.

$$E_\mu[W] = e^{-\mu^2 t/2\sigma^2} E_0 \left[W e^{\mu X(t)/\sigma^2} \right]$$