Homework Problem Notes: Chapters 6-7

6.2 In this game of betting, the return vector takes on a special form: o_i for outcome i and -1 for all other outcomes. The Arbitrage Theorem applied to this situation yields the condition that there will be no arbitrage if

$$\sum_{i=1}^{4} \frac{1}{1+o_i} = 1.$$

All the values o_i are known except o_4 , so the above condition becomes an equation for o_4 and the solution is 47/13.

6.3 There are two ways to solve this problem depending on whether you use part (a) or part (b) of the Arbitrage Theorem. If you suspect that there are arbitrage opportunities and can find just one instance with particular x values that lead to positive returns in all outcomes, then part (b) should be the easy way to go. On the other hand, if you see that there will be no solution to the p vector in part (a) that satisfies the positive and sum one conditions, then you can also claim arbitrage.

In part (b) of the problem, we can solve for the p vector in terms of a free coefficient and we find that p components would have the same sign. This suggests that by choosing x appropriately we can end up with a p vector that satisfies the conditions aforementioned. That solution turns out to be x = -90.

6.6 The determination of no arbitrage cost of the option is always based on the discounted expectation of the payoff in the *risk-neutral* world, namely by using the risk-free rate as the expected growth rate of the underlying stock.

$$P = \frac{1}{1+r} \mathbb{E}\left[(K - S(T))^+ \right] = \frac{1}{1+r} \left[p \cdot 0 + (1-p)50 \right] = \frac{200(1-r)}{3(1+r)}$$

6.7 First we determine the risk-neutral probability measure,

$$p = \frac{1+r-d}{u-d} = \frac{1+2r}{3}$$

Then the no arbitrage cost of the option

$$C = \frac{1}{(1+r)^2} \mathbb{E} \left[(S(2) - 150)^+ \right]$$

= $\frac{1}{(1+r)^2} \left[250p^2 + 0 \cdot p(1-p) + 0 \cdot (1-p)^2 \right]$
= $\frac{250(1+2r)^2}{9(1+r)^2}$

6.10 We want to consider a general option where the payoff is specified as

$$payoff = \begin{cases} a & \text{if } S(1) = uS(0) \\ b & \text{if } S(1) = dS(0) \end{cases}$$

Let C be the no arbitrage cost of the option. We show that it can be replicated by observing that we can buy m shares of the stock, and deposit C - mS(0) in the bank, and after one time period the payoff of the portfolio (stock + bank deposit) is

$$payoff = \begin{cases} muS(0) + (C - mS(0))(1 + r) & \text{if } S(1) = uS(0) \\ duS(0) + (C - mS(0))(1 + r) & \text{if } S(1) = dS(0) \end{cases}$$

The matching of the payoffs require a solution to a linear system, which gives

$$m = \frac{a-b}{S(u-d)}$$

which tells us how many shares to buy or sell (if m < 0).

6.11 We realize that the condition for a nonzero payoff, that the price decreases in the first two periods, is the same as S(2) < 100. The only path that leads to a positive payoff is $S(0) = 100 \rightarrow S(1) = 80 \rightarrow S(2) = 64 \rightarrow S(3) = 80 \rightarrow S(4) = 100 \rightarrow S(5) = 125$, which gives a payoff of 25. The no arbitrage price of the option is therefore

$$C = \frac{25}{(1+r)^5} (1-p)^2 p^3 \approx 0.511$$

7.1 Note

$$\frac{S(n)}{S(n-1)} = \exp\left(X(n) - X(n-1)\right) = \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}Z\right)$$

So

$$\log\left(\frac{S(n)}{S(n-1)}\right) = (\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}Z$$

The standard deviation is $\sigma \sqrt{\Delta t}$. We can use $\Delta t = 1/252$ for daily changes, or $\Delta t = 1/52$ for weekly changes.

7.2 The probability to calculate is

$$P\left\{S(1/3) > 42\right\} = P\left\{\log\frac{S(1/3)}{S(0)} > \log\frac{42}{40}\right\} = 0.475$$

We note that here we use the actual expected growth rate $\mu = 0.12$.

7.5 (a) This is similar to 7.2, where we need to calculate the probability in the actual world with $\mu = 0.06$.

$$P\left\{\frac{S(1/2)}{S(0)} < 0.9\right\} = 0.262$$

(b) We should solve

$$A = e^{-0.05 \times 0.5} \tilde{\mathbb{E}}[100I_{S(1/2) < 0.9S(0)}] = 100e^{-0.05 \times 0.5} \tilde{P} \{S(1/2) < 0.9S(0)\} = 29.8$$

We use \tilde{P} here because this is under the risk-neutral probability measure, with $r - \frac{1}{2}\sigma^2$ used.

- 7.6 (a) Straightforward use of the Black-Scholes formula leads to C = 3.96.
 - (b) Here we use the probability in the actual world.

$$P\left\{S\left(\frac{1}{4}\right) < 100\right\} = 0.602$$

(c) Here we use the independence property of the Brownian motion

$$P\left\{S(0.5) \ge 105, S(1) \ge S(0.5)\right\} = P\left\{S(0.5) \ge 105\right\} \cdot P\left\{S(1) \ge S(0.5)\right\}$$

But we determine the no arbitrage cost of an option, we must use the riskneutral probability measure, which means the value $r - \frac{1}{2}\sigma^2 = -0.005$ must be used instead of $\mu = 0.05$ for the drift parameter.

$$C = 50e^{-0.04}\tilde{P}\left\{S(0.5) \ge 105\right\} \cdot \tilde{P}\left\{S(1) \ge S(0.5)\right\} = 7.48$$

7.10 We should solve the equation for x

$$10 = e^{-0.06} \left[5\tilde{P} \left\{ S(1) < 95 \right\} + x\tilde{P} \left\{ S(1) > 110 \right\} \right]$$

In this calculation, we must use $r - \frac{1}{2}\sigma^2 = -0.02$ for the growth rate in the riskneutral world, and the solution is x = 24.46.

7.12 The equation for x is

$$10 = 100e^{-0.02}\tilde{P}\left\{S(1) > (1+x)40\right\}$$

(a) Again we must use the drift parameter $r - \frac{1}{2}\sigma^2 = 0$ in the calculation:

$$1 - \Phi\left(\frac{\log(1+x)}{0.2}\right) = \frac{e^{0.02}}{10}$$

So x = 0.289.

(b) In the actual probability measure, we use the drift parameter $\mu = 0.04$,

$$P\left\{S(1) > 40(1+x)\right\} = P\left\{\log\left(\frac{S(1)}{S(0)}\right) > \log(1+x)\right\} = 1 - \Phi\left(\frac{\log 1.289 - 0.04}{0.2}\right) = 0.1425$$