Math 5610/6860, Sample Test

1. Find the rate of convergence of the following sequences as $n \to \infty$ or $h \to 0$.

   (a) $\lim_{n\to\infty} \left(1 - \cos\frac{1}{n}\right) = 0$

   (b) $\lim_{h\to 0} \frac{1-e^h}{h} = -1$

2. Consider the equation $f(x) = x - \frac{1}{2}e^{-x} = 0$.

   (a) We want to use the fixed-point iterations based on $g(x) = \frac{1}{2}e^{-x}$, suggest an interval for picking initial guess p_0 that will guarantee convergence and justify.

   (b) Write down the iteration formulas for Newton's method and secant method.

3. Show that the sequence $p_n = 10^{-\alpha^n}$ for $\alpha > 1$ converges to 0 with order α.

4. Derive the system of equations for the cubic spline coefficients with three nodes $x_0 = 0, x_1 = 0.1$, and $x_2 = 0.2$, and corresponding function values. The cubic spline is free at the left end but clamped at the right end with $f'(0.2)$ given.

5. Using the function values of f at $x = 0, h$, and $2h$, derive a second-order finite difference approximation to $f'(0)$.

6. Suppose we want to use numerical quadratures to approximate $\int_{-1}^{1} |x| dx$, and we consider the following choices: (a) composite trapezoidal with $n = 2$, (b) Simpson's rule. Which one would you suggest? Compute both quadratures and explain why you would prefer one to the other.