Solutions for the Final Practice

1. (a) $e^{0.01} \approx 1.01$, $\cos(0.01) \approx 1.00$, $\sin(0.01) \approx 0.01$

$$\frac{e^{0.01} - \cos(0.01)}{\sin(0.01)} \approx \frac{1.01 - 1.00}{0.01} = 1$$

(b) $e^x = 1 + x + \frac{1}{2}x^2 + \cdots$

$\cos x = 1 - \frac{1}{2}x^2 + \cdots$

$\sin x = x \cancel{+} \frac{1}{3!}x^3 + \cdots$

so $e^x - \cos x = x + x^2 + \cdots$

$$\frac{e^x - \cos x}{\sin x} = 1 + x + \cdots$$

(c) Plugging in $x = 0.01$, $f(0.01) \approx 1 + 0.01 + \cdots \approx 1.01$

This is much closer to the exact value than 1.

2. (a) $f'(x) = \frac{1}{1+x} - 1 = -\frac{x}{1+x} < 0$ for $x \in [0.5, 1.5]$

$g'(x) > 0$ for $x \in [0.5, 1.5]$

$g(0.5) = \cancel{0.155}\ \cancel{1.155}, 0.655$, $g(1.5) = 1.166$

So $g(x)$ maps $[0.5, 1.5]$ into $[0.5, 1.5]$

and $|g(x)| = \left|\frac{1}{1+x}\right| \le 0.667$ for $x \in [0.5, 1.5]$

Based on theorem 2.3, $g(x)$ has a fixed point in $[0.5, 1.5]$

(b) $k = 0.667$

$$|p - p_n| \le \frac{k^n}{1-k}|p_1 - p_0| \le 10^{-6}$$

$p_1 = g(p_0) = \log 2. + 0.25 \approx 0.943$

So we need

$$k^n \le (1-k)\frac{10^{-6}}{|p_1 - p_0|} \approx 5.84 \times 10^{-6}$$

or $n \geq 30$

3. $S(x) = ax^3 + bx^2 + cx + d$

$S(0) = 1 \Rightarrow d = 1$

$S''(0) = 0 \Rightarrow b = 0$

$$\left.\begin{array}{l} S(1) = 0 \Rightarrow a + c + 1 = 0 \\ S'(1) = 0 \Rightarrow 3a + c = 0 \end{array}\right\} \quad \begin{array}{l} a = 1/2 \\ c = -3/2 \end{array}$$

$$\Rightarrow S(x) = \frac{1}{2}x^3 - \frac{3}{2}x + 1$$

4. Replace h by $h/2$

$$f'(x) = N_0(h/2) - \frac{h}{4}f''(x) - \frac{h^2}{4 \cdot 3!}f'''(x) - \cdots$$

Subtract the original by 2× (the above)

$$-f'(x) = \underbrace{N_0(h) - 2N_0\left(\frac{h}{2}\right)}_{-N_1(h)} + O(h^2)$$

$$N_1(h) = 2N_0(h/2) - N_0(h)$$

$$= 2\,\frac{f(x+h/2) - f(x)}{h/2} - \frac{f(x+h) - f(x)}{h}$$

$$= \frac{1}{h}\left[4f\left(x+\frac{h}{2}\right) - 3f(x) - f(x+h)\right]$$

5. $x = \frac{t+1}{2}\pi$: maps $[-1, 1]$ to $[0, \pi]$

$$\int_0^\pi x\cos x\,dx = \frac{\pi}{2}\int_{-1}^{1}\left(\frac{t+1}{2}\pi\right)\cos\left(\frac{t+1}{2}\pi\right)dt$$

$$= \left(\frac{\pi}{2}\right)^2\int_{-1}^{1}(t+1)\cos\left(\frac{t+1}{2}\pi\right)dt$$

$$= \left(\frac{\pi}{2}\right)^2\int_{-1}^{1}f(t)\,dt$$

Using Gaussian quadrature with $n = 3$

$$\int_{-1}^{1} f(t)\,dt \approx 0.556\,(f(0.7446) + f(-0.7446)) + 0.889\,f(0)$$

$$\approx -0.8079$$

$\int_0^{\pi} x\cos x\,dx \approx -1.993$, compare this with the exact answer (-2).

6. Define $u_1 = y$, $u_2 = y'$.

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}' = \begin{bmatrix} u_2 \\ f(t) - p(t)\,u_2 - g(t)\,u_1 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

Here we use the midpoint method: $\left(w^{(1)} \approx u_1,\ w^{(2)} \approx u_2 \right)$

$$w_{n+1}^{(1)} = w_n^{(1)} + h\, f_1\left(t_n + \tfrac{\Delta t}{2},\ w_n^{(1)} + \tfrac{h}{2} f_1(t_n, w_n)\right)$$

$$w_{n+1}^{(2)} = w_n^{(2)} + h\, f_2(\qquad \text{"} \qquad)$$

where

$$f_1(\text{"}) = w_n^{(2)} + \tfrac{h}{2}\left[f(t_n) - p(t_n)\,w_n^{(2)} - g(t_n)\,w_n^{(1)}\right]$$

$$f_2(\text{"}) = f\left(t_n + \tfrac{h}{2}\right) - p\left(t_n + \tfrac{h}{2}\right) \cdot \left[w_n^{(2)} + \tfrac{h}{2}\left(f(t_n) - p(t_n)\,w_n^{(2)} - g(t_n)\,w_n^{(1)}\right)\right] - g\left(t_n + \tfrac{h}{2}\right) \cdot \left[w_n^{(1)} + \tfrac{h}{2}\,w_n^{(2)}\right]$$

7. $$\hat{y} = -1.75 + 1.8929\,x$$

8. $$J = \begin{bmatrix} 2x & -4y \\ 3\cos 3x & 1 \end{bmatrix}, \quad J^{-1} = \frac{1}{\det J} \begin{bmatrix} 1 & 4y \\ -3\cos 3x & 2x \end{bmatrix}$$

$$\begin{bmatrix} x^{(k)} \\ y^{(k)} \end{bmatrix} = \begin{bmatrix} x^{(k-1)} \\ y^{(k-1)} \end{bmatrix} - \frac{1}{\det J(x^{(k-1)}, y^{(k-1)})} \begin{bmatrix} 1 & 4y^{(k-1)} \\ -3\cos 3x^{(k-1)} & 2x^{(k-1)} \end{bmatrix} \cdot \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$f_1 = (x^{(k-1)})^2 - 2(y^{(k-1)})^2, \quad f_2 = y^{(k-1)} - \sin 3x^{(k-1)}$$