Math 5610/6860 Final Practice Problems

1. Let
$$f(x) = \frac{e^x - \cos x}{\sin x}$$
   (a) Use 3-digit rounding arithmetic to evaluate $f(0.01)$.

   (b) An alternative to evaluate $f(0.01)$ is to use Taylor series expansions of the three functions involved $(\cos(x), \sin(x), e^x)$ to replace various terms in f but retain only terms up to $O(x^2)$, then evaluate the resulting approximation using the same three-digit rounding arithmetic.

   (c) Show that even with Taylor series expansion approximations, the second approach results in substantial improvement over the first by comparing with the exact value.

2. We try to find a solution of equation $f(x) = \log(1+x) + 0.25 - x = 0$ in $[0.5, 1.5]$. The obvious choice of iteration is
$$g(x) = \log(1+x) + 0.25.$$
   (a) Show that for any $x \in [0.5, 1.5]$, $g(x) \in [0.5, 1.5]$, therefore there is a fixed point in $[0.5, 1.5]$.

   (b) Find a positive constant $k < 1$ such that $g'(x) \le k$ for all $x \in [0.5, 1.5]$, then estimate the maximum number of iterations needed if we start with $p_0 = 1$ and require that the absolute error of the approximation is less than 10^{-6}.

3. Find a cubic spline $S(x)$ for $x \in [0,1]$ satisfying $S(0) = 1, S(1) = 0, S''(0) = S'(1) = 0$.

4. From the Taylor expansion
$$\frac{f(x+h) - f(x)}{h} = f'(x) + \frac{h}{2} f''(x) + \frac{h^2}{3!} f'''(x) + \dots$$
we have
$$f'(x) = N_0(h) - \frac{h}{2} f''(x) - \frac{h^2}{3!} f'''(x) - \dots$$
Use Richardson's extrapolation to derive the next approximation $N_1(h)$ for $f'(x)$. What is the order of accuracy for N_1? Do you recognize this approximation for $f'(x)$?

5. Use Gaussian quadrature with $n = 3$ to approximate
$$\int_0^\pi x \cos x \, dx.$$

6. Consider the second-order ODE for $y(t)$
$$y'' + p(t)y + q(t)y = f(t)$$
where $p(t)$, $q(t)$ and $f(t)$ are given, and the initial conditions $y(0) = 0$, $y'(0) = \alpha$. Cast the problem in the form of a system of first-order equations and suggest a second-order method to solve it numerically. Explicitly write down the iteration formulas for $w_n \approx y(t_n)$, where $t_n = n\Delta t, n = 0, 1, \dots, N$.

7. Find the linear least square approximation for the data points

x_i	1	2	4
y_i	0.5	1.5	6

8. Consider the nonlinear system

$$\begin{aligned} x^2 - 2y^2 &= 0 \\ y - \sin 3x &= 1 \end{aligned}$$

Write down the iteration steps from Newton's method, assuming an initial guess (x_0, y_0).