Math 5610/6860 Final Study Sheet

This is a comprehensive final so most materials discussed in the semester can be included, unless specifically excluded. We covered the following seven chapters: Chapters 1-5, chapter 8 and part of chapter 10. In the following, we give a summary based on chapter by chapter listing.

Chapter 1:

Preliminaries involving calculus and computer arithmetic

- Taylor's theorem and Taylor series expansion are probably the most important calculus tool we need in this semester, especially in error analysis and construction of improved approximations.
- You should also get familiar with the floating number representation, machine epsilon and rounding/chopping errors.
- The concept of convergence is formerly introduced and other related definitions such as *rate, or order, of convergence* and *converge to p of order α* should be distinguished. The terminology can be confusing and it is better understood in the context of the subject. The other notion that should be emphasized is little-oh and big-Oh notations which are conveniently used to avoid unnecessary details.
- Some common tricks to turn an expression into another equivalent one, with potential significant improvements in accuracy due to round-off errors.

Chapter 2:

We discussed basic methods to solve a single equation with a single variable. The equation is always nonlinear (otherwise it's too trivial and does not require a numerical method) so we should always be aware of the existence/uniqueness issues.

- Bisection method: only linear convergence, but in some situations more reliable.
- Fixed-point iteration contains a class of methods, denoted by the mapping $g(x)$ that comes out of $f(x) = 0$. For each suggested $g(x)$, we need to verify these two conditions in Fixed-Point Theorem (2.4), which depend on the crucial choice of the interval $[a, b]$.
- Newton's method can be viewed as one particular fixed-point iteration, though it has roots in approximating a function locally by a linear function. The quadratic convergence is noted for this method, even though the choice of initial guess is essential for the method to be successful.
- Secant method is based on Newton's method with its focus on simplifying the evaluation of derivative $f'(x)$. These modifications usually reduce the cost of the iteration at the expense of losing quadratic convergence.
- Aitken's acceleration, Steffensen's method, and Muller's method will not be covered in the final.

Chapter 3:

A classic topic in numerical analysis, but has lost some of its glories since the advent of modern computations.

- Lagrange interpolation: easy to write down but tedious and impractical to use. It's important to understand the construction. The error form is an important tool to do error analysis.
- Newton's divided differences: nice to work with on the spreadsheet, but probably not as useful as in its glorious days.
- Hermite interpolation: more controlled than Lagrange polynomials but require additional information.
- Cubic splines: *the* method of choice for many applications. You should know all the continuity conditions and boundary conditions. You should also be familiar with the construction and reduction of the system for coefficients.

Chapter 4:

- Finite difference approximation of the derivatives is based on Taylor series expansion again. Order of accuracy should be clearly defined, usually more points result in better approximation, except in the centered difference case where you can get one extra order for free if you choose the locations of the points right.
- Richardson's extrapolation provides a systematic way to improve approximation in either numerical differentiation and numerical quadrature. It is important to know the form of the error though and the actual coefficients in the expansion are less crucial.
- Definition of numerical quadrature as a finite sum. Accuracy defined by degree of accuracy measured by its performance on polynomials.
- Basic quadratures: trapezoidal, Simpson's, midpoint, etc.
- Gaussian quadrature: with the same number of points, doubles the degree of accuracy compared to others. The downside is that you need to choose the points according to the table. This limits its applicability in practice.
- Multiple integrals and improper integrals: natural extensions of the above techniques. The improper integrals usually require special care in that some procedures should be used before discretization.

Chapter 5:

Some theory is required before attempting any numerical methods. The concepts of Lipschitz continuity and Lipschitz constant are crucial in the initial value problems.

- Euler's method is the father to all other methods and can be still useful in practice. The truncation error and stability can be better understood in this context.

- High-order methods are divided into two categories: those based on Taylor's expansion, and the Runge-Kutta type, the latter is probably more useful.

- Most practical methods found in software packages are based on variable step adaptive methods. You should know the main considerations in choosing discretization parameters depending on the problem and different sections.

- Multistep methods can be powerful and economical, if the stability issues are taken care of. We need to know the basic root condition.

- High-order equations and system of equations: first we extend all the methods to systems of equations, then for high-order equations, there is a way to turn them into systems and you should certainly be able to convert accordingly.

- Stiff equations: need to know the definition of stiffness. Typically an implicit method is needed to solve the problem.

Chapter 8:

The approximation theory in practice serves two purposes: to represent a function that is otherwise known only at a finite set of points, and to replace an otherwise expensive function with a more use-friendly function.

- Discrete least square approximation: probably the most popular with scientists in various fields with a lot of data sets. The normal equation is simple to form and we should know how it is derived and be able to solve for the coefficients.

- Orthogonal polynomials: we should know the benefit provided by orthogonal polynomials. It is useful to find the connection between two functions orthogonal to each other and two vectors perpendicular to each other. An extra factor is the weight function in orthogonal function definition.

- Chebyshev polynomials: we need to know the particular advantages of these polynomials, such as minimum maximum absolute value property.

- Rational function approximations: provide an alternative to polynomial approximations with somewhat better behavior. The determination of coefficients can be rather expensive.

- Trigonometric polynomial approximation leads to the famous Fourier series and it has never lost its popularity. One of the properties is the minimum least square error. The determination of coefficients involve computations of integrals which can be expensive to evaluate. The discrete version is even more useful and the integrals in coefficients are replaced by finite sums.

- FFT is a fast way to compute these coefficients in the discrete trigonometric polynomial interpolation ($m = n$). In practice the function in software packages such as `fft/ifft` should be used. We should be able to sketch the counting of numbers of operations to appreciate the reduction from $O(N^2)$ to $O(N \log N)$.

Chapter 10:

We only emphasize the standard Newton's method and one particular quasi-Newton method (Broyden's method). The focus is the Jacobian matrix and how we can get around it by either replacing with finite difference approximation, or even better, coming up with some approximation that can be updated in iterations.