Test No.1, Math 3150-3, September 13, 2007

Name: ____________________ **U. ID:** ____________

Instructions: This is a closed book, but open written notes test. Calculators are not allowed. You need to show all the details of your work to receive full credits.

Problem	1	2	3	4	total
worth of points	25	25	25	25	100
your points					

1. Verify that the function

$$u(x,t) = \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4t}}$$

is a solution of the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \ t > 0.$$

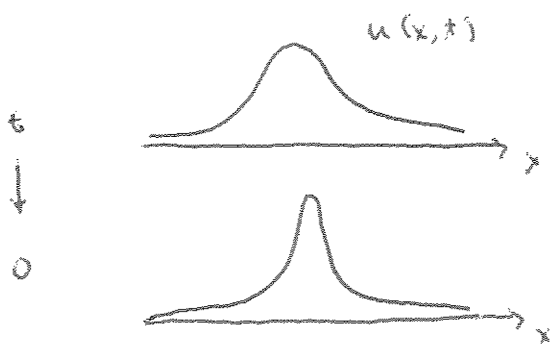
What willl happen to u when $x \to \pm\infty$? Show the tendency of u as functions of x when $t \to 0+$, by sketching snapshots of u for several diminishing values of t.

$$\frac{\partial u}{\partial t} = \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4t}} \cdot \left(\frac{x^2}{4t^2}\right) - \frac{1}{2t^{3/2}} e^{-\frac{x^2}{4t}} = \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4t}} \left[\frac{x^2}{4t^2} - \frac{1}{2t}\right]$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4t}} \left(-\frac{x}{2t}\right), \quad \frac{\partial^2 u}{\partial x^2} = -\frac{1}{2t^{3/2}}\left[e^{-\frac{x^2}{4t}} \cdot x\left(-\frac{x}{2t}\right) + e^{-\frac{x^2}{4t}}\right]$$

$$= \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4t}} \left(\frac{x^2}{4t^2} - \frac{1}{2t}\right) = \frac{\partial u}{\partial t}$$

$u \to 0$ as $x \to \pm\infty$ as $t > 0$ fixed



As $t \to 0+$, the profile becomes more and more concentrated around $x=0$

they converge to the Dirac delta function.

2. Determine for each of the following functions whether it is periodic, whether it is piecewise continuous, and whether it is even, odd, or neither. In the case of periodic functions, also determine the fundamental period.

   (a) $f(x) = \cos^3 x$

   periodic, piecewise continuous, even

   $T = 2\pi$

   (b) $f(x) = \cos(2x^3)$

   non periodic, piecewise continuous even

   (c) $f(x) = (x^3 + \sin x)^4$

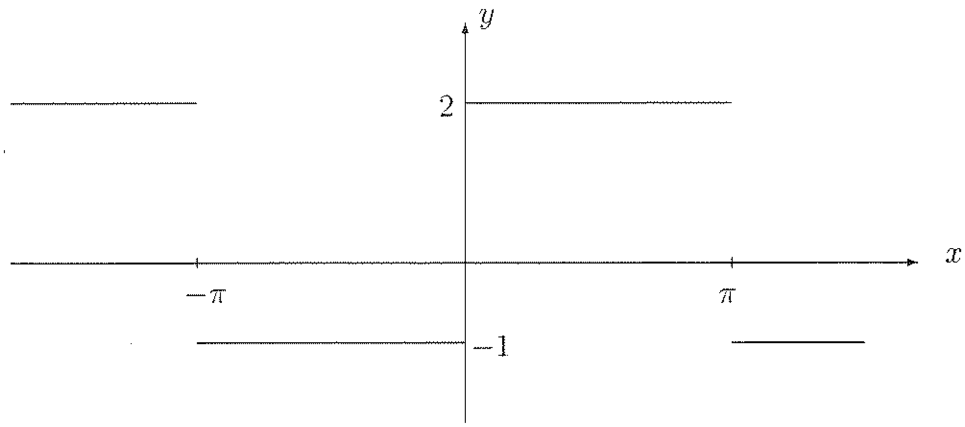
   non periodic, piecewise continuous even

   (d) $f(x) = \tan|x|$

   non periodic, not piecewise continuous

   even

3. Find the Fourier series of the 2π-periodic function plotted below.



$$a_0 = \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-1)\,dx + \int_{0}^{\pi} 2\cdot dx\right] = \tfrac{1}{2}$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\cos nx\,dx + \int_{0}^{\pi} 2\cos nx\,dx\right] = \frac{1}{\pi}\left[0 + 0\right] = 0$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\sin nx\,dx + \int_{0}^{\pi} 2\sin nx\,dx\right] = \frac{1}{\pi}\left[\frac{1}{n}(1-\cos n\pi) - \frac{2}{n}(\cos n\pi - 1)\right]$$

$$= \frac{3}{n\pi}(1-\cos n\pi) = \begin{cases} 0 & n \text{ even} \\ \frac{6}{n\pi} & n \text{ odd} \end{cases}$$

$$f(x) = \frac{1}{2} + \frac{6}{\pi}\sum_{k=0}^{\infty} \frac{\sin(2k+1)x}{2k+1}$$

4. Find the cosine expansion of the function

$$g(x) = \sin x, \quad 0 < x < \pi.$$

Comment on the relationship between this expansion and the Fourier series for the 2π-periodic function

$$f(x) = |\sin x|, \quad \text{if } -\pi < x < \pi.$$

The cosine expansion of $g(x)$ should be exactly the same as the Fourier series for $f(x)$:

$$\frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k)^2 - 1} \cos 2kx$$

$$a_0 = \frac{1}{\pi} \int_0^{\pi} \sin x \, dx = -\frac{1}{\pi} \left[\cos \pi - \cos 0 \right] = \frac{2}{\pi}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} \sin x \cos nx \, dx = \frac{2}{\pi} \left[\frac{\cos(n-1)x}{2(n-1)} - \frac{\cos(n+1)x}{2(n+1)} \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{\cos(n-1)\pi - 1}{n-1} - \frac{\cos(n+1)\pi - 1}{n+1} \right] \quad \text{if } n \neq 1$$

$$= \begin{cases} 0 & n \text{ odd} \\ -\frac{4}{\pi} \frac{1}{n^2 - 1} & \end{cases}$$

$$a_1 = \frac{2}{\pi} \int_0^{\pi} \sin x \cos x \, dx = 0$$