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LOOP GROUPS AND EQUATIONS OF KdV TYPE

by GRAEME SEGAL and GEORGE WILSON

The purpose of this paper is to work out some of the implications of recent ideas of M. and Y. Sato about the Korteweg-de Vries (KdV) equation and related non-linear partial differential equations. We learned of these ideas from the papers [5] of Date, Jimbo, Kashiwara and Miwa (the original work of M. and Y. Sato appears to be available only in Japanese). We shall describe a construction which assigns a solution of the KdV equation to each point of a certain infinite dimensional Grassmannian. The class of solutions obtained in this way, which is misleadingly referred to as " the general solution " in [5], includes the explicit algebro-geometric solutions of Krichever [10, 11]; among these are the well known " n-soliton " and rational solutions.

Our main aims are to determine what class of solutions is obtained by the method, to illustrate in detail how the geometry of the Grassmannian is reflected in properties of the solutions, and to show how the algebro-geometric solutions fit into the picture. We have also tried to explain the geometric meaning of the " τ-function ", which plays a fundamental role in the papers [5]. But above all we have endeavoured to present a clear and self-contained account of the theory, and hope to have elucidated a number of points left obscure in the literature.

1. Introduction

The KdV equation

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + 6u\,\frac{\partial u}{\partial x}$$

describes the time-evolution of a function u of the variable x: we think of the equation geometrically as defining a flow on a suitable space of functions u. It is well known that the theory of the equation is closely connected with that of the linear differential operator $L_u = D^2 + u$, where $D = \partial/\partial x$, which is to be regarded as an operator on functions of x which varies with time. In fact the KdV equation can be written in the " Lax form "

$$\frac{\partial L_u}{\partial t} = 4[P_u, L_u],$$

where P_u is the operator $D^3 + \dfrac{3}{4}(uD + Du)$.

The operator P_u is almost characterized by the fact that—for any function u—the commutator $[P_u, L_u]$ is a multiplication operator. More precisely, for given L_u there is a canonical sequence of operators

$$P_u^{(k)} = D^k + p_{k2} D^{k-2} + \dots + p_{kk}$$

such that each $[P_u^{(k)}, L_u]$ is a multiplication operator, and any operator P with the same property is a constant linear combination of the $P_u^{(k)}$. It turns out that the coefficients of such an operator P must be differential polynomials in u, i.e. polynomials in u and its x-derivatives $\partial^j u/\partial x^j$. For each k the equation

$$\frac{\partial L_u}{\partial t} = [P_u^{(k)}, L_u] \tag{1.1}$$

defines a flow on the space of functions of x. These flows are called the "KdV hierarchy". The case $k = 3$ is the original KdV equation (apart from the factor 4). When $k = 1$ we have $P_u^{(1)} = D$, and the corresponding flow is just uniform translation of u. When k is even we have $P_u^{(k)} = (L_u)^{k/2}$, so that the corresponding flow is stationary. It is a fundamental theorem of the subject that the flows given by (1.1) for various k *commute* among themselves.

In this paper we shall describe the KdV flows on a certain class $\mathscr{C}^{(2)}$ of functions u. Our approach is in terms of the geometry of an infinite dimensional manifold which is of considerable interest in its own right. It has two alternative descriptions. The first is as the space ΩU_2 of loops in the unitary group U_2. The second, more immediately relevant, description is as the Grassmannian $Gr^{(2)}$ of all closed subspaces W of the Hilbert space $H = L^2(S^1)$ of square-summable complex-valued functions on the circle $S^1 = \{z \in \mathbf{C} : |z| = 1\}$ which satisfy the two conditions

(i) $z^2 W \subset W$, and
(ii) W is comparable with H_+.

Here z denotes the operator $H \to H$ given by multiplication by the function z on S^1, and H_+ is the closed subspace of H spanned by $\{z^k\}$ for $k \geq 0$, i.e. the boundary values of holomorphic functions in $|z| < 1$. The meaning of "comparable" is explained in § 2.

Our basic construction associates to each point W in a connected component of $Gr^{(2)}$ a meromorphic function u_W on the line, belonging to the class $\mathscr{C}^{(2)}$. The group Γ_+ of holomorphic maps $D_0 \to \mathbf{C}^\times$, where D_0 is the disc $\{z \in \mathbf{C} : z \leq 1\}$, acts by multiplication operators on H, and hence acts on $Gr^{(2)}$. The action of Γ_+ induces the KdV flows on $\mathscr{C}^{(2)}$ in the following sense: if

$$g = \exp \Sigma t_k z^k \in \Gamma_+,$$

where $(t_1, t_2, \dots)$ are real numbers almost all zero, then u_{gW} is the function obtained from u_W by letting it flow for time t_k along the k-th KdV flow, for each k. (This makes sense precisely because the KdV flows commute.)

The meromorphic function u_W is obtained from the so-called "τ-function" τ_W of W by the formula

$$u_W(x) = 2\left(\frac{d}{dx}\right)^2 \log \tau_W(x);$$

$\tau_W(x)$ is the determinant of the orthogonal projection $e^{-xz}W \to H_+$. Of course the determinant needs to be suitably interpreted. To define it one must choose bases in W and H_+, and accordingly $\tau_W(x)$ is defined only up to a multiplier independent of x. The determinant $\tau_W(x)$ vanishes, and hence $u_W(x)$ has a pole, precisely when $e^{-xz}W$ intersects $H_+^\perp$.

For certain particular subspaces W belonging to the Grassmannian it turns out that the τ-function is a Schur function. This was discovered by Sato, and it was, we have been told, the observation that led him to develop his theory. In general a point of the Grassmannian can be described by its Plücker coordinates, and (as we shall prove in § 8) the corresponding τ-function is an infinite linear combination of Schur functions with the Plücker coordinates as coefficients.

It is not practical, however, in developing the theory, to pass directly from the τ-function to the function u_W. Instead, one introduces an intermediate object, the *Baker function* ψ_W. This is an eigenfunction of the operator $D^2 + u_W$:

$$(D^2 + u_W)\,\psi_W(x, z) = z^2\,\psi_W(x, z);$$

on the other hand for each fixed x it is the unique element of W which is of the form

$$e^{xz}(1 + a_1(x)\,z^{-1} + a_2(x)\,z^{-2} + \dots). \tag{1.2}$$

Finding the formula ((5.14) below) for the Baker function in terms of the τ-function was one of the most important contributions of the Japanese school. The formula is a precise analogue of a formula known earlier, in the case of solutions arising from an algebraic curve, expressing the Baker function in terms of θ-functions.

At this point we should say something about the class $\mathscr{C}^{(2)}$ of solutions u_W which we obtain. Suppose to begin with that u is a C^∞ function defined in an interval of $\mathbf{R}$. Then the eigenvalue problem $L_u\,\psi = z^2\,\psi$ has a *formal* solution of the form (1.2). The coefficients a_i in the formal series are C^∞ functions determined recursively by

$$-2a_i' = L_u\,a_{i-1},$$

with $a_0 = 1$. Each successive a_i involves a new constant of integration: this means that ψ is determined up to multiplication by an arbitrary power series in z^{-1} with constant coefficients. The series (1.2) will usually not converge for any values of z. The class of functions $\mathscr{C}^{(2)}$ is, roughly speaking, those such that it can be chosen convergent in a neighbourhood of $z = \infty$. To see how restrictive this is, consider the case of functions u which are rapidly decreasing as $x \to \pm\infty$. Then there are unique genuine solutions $\psi_+(x, z)$ and $\psi_-(x, z)$ of $L_u \psi = z^2 \psi$, defined and holomorphic in z for $Re(z) > 0$ and $Re(z) < 0$ respectively, characterized by the properties

$$\psi_+(x, z) \sim e^{xz} \quad \text{as } x \to -\infty,$$
$$\psi_-(x, z) \sim e^{xz} \quad \text{as } x \to +\infty.$$

These solutions both extend to the axis $Re(z) = 0$, but unless u belongs to the exceptional class of so-called "reflectionless potentials" or "multisolitons" they will be linearly independent functions of x, and then no genuine solution of the form (1.2) can exist. The situation is similar if we consider the case where u is a real C^∞ periodic function: of these, our class $\mathscr{C}^{(2)}$ contains only the "finite gap" potentials u. The periodic KdV flows have been described by McKean and Trubowitz [25] in terms of Riemann surfaces of (in general) infinite genus: the finite gap potentials are precisely those for which the Riemann surface involved is of finite genus. The corresponding solutions to the KdV hierarchy are then included in the class obtained by Krichever's method.

We next explain how Krichever's construction is included in ours. Krichever associates a function of x, say $u_{X,\mathscr{L}}$, to an algebraic curve X with a distinguished point x_∞ and a line bundle $\mathscr{L}$ (and also some additional data which we shall overlook in this introduction). A solution of the KdV equation is obtained by letting $\mathscr{L}$ move along a straight line in the Jacobian of X. We shall see that a space $W \in Gr^{(2)}$ is naturally associated to Krichever's data. Think of the circle S^1 as a small circle around the point x_∞ of X; then W consists of those functions on S^1 which are boundary values of holomorphic sections of $\mathscr{L}$ outside S^1—we suppose that $\mathscr{L}$ has been trivialized near x_∞. Krichever's solution $u_{X,\mathscr{L}}$ is simply u_W. When the curve X is non-singular, we shall show in § 9 that the τ-function τ_W is essentially the θ-function of X.

The algebro-geometric solutions u are precisely those such that the operator L_u commutes with an operator of odd order. There is a very elegant theory, due essentially to Burchnall and Chaundy [4], relating commutative rings of differential operators to algebraic curves. A modern treatment of the subject has been given by Mumford [16]; but as it fits very naturally into our framework we have included a short self-contained account in § 6.

We shall describe in particular detail the KdV flows on the two dense subspaces $Gr_0^{(2)}$, and $Gr_1^{(2)}$ of the Grassmannian corresponding respectively to polynomial and rational loops in U_2. The first space corresponds exactly to the rational solutions of the KdV equations which are zero at ∞. It is a beautiful fact that the orbits of the group Γ_+ of KdV flows on $Gr_0^{(2)}$ form a cell decomposition of $Gr_0^{(2)}$, with one cell of each complex dimension. (The n-th cell is the orbit of the function $-n(n+1)/x^2$.)

The points of $Gr_1^{(2)}$ are those that arise by Krichever's construction from rational curves with singularities. For any $W \in Gr_1^{(2)}$ the orbit of W under Γ_+ can be identified with the Jacobian of the corresponding curve.

The KdV hierarchy has fairly obvious generalizations in which the operator

$D^2 + u$ is replaced by an operator of order n: these hierarchies are related to the loop space of U_n in the same way that the KdV equations are related to ΩU_2. For simplicity of explanation we have restricted ourselves in the introduction to the case $n = 2$, but in the body of the paper we shall always treat the general case, which presents no additional difficulty. In fact we shall treat a more general hierarchy still, that of the "Kadomtsev-Petviashvili" (KP) equations; the hierarchies already mentioned are all specializations of this. Less obvious are the generalizations of the KdV hierarchy due to Drinfel'd and Sokolov [6], in which, roughly speaking, U_n is replaced by an arbitrary compact Lie group; more precisely, Drinfel'd and Sokolov associate several "KdV" hierarchies to each affine Kac-Moody algebra. Some of these hierarchies are discussed in [5], though no general theory is developed there. The key step in [6] which is missing from [5] is to view the KdV flows as quotients of certain simpler ones, the "modified KdV" flows [12, 20]: the generalization of the latter is fairly evident. We refer to [35] for a brief account of how the present theory generalizes to the equations of Drinfel'd and Sokolov: here we just mention that from this point of view our main construction appears as a special case of a well known procedure ("dressing") of Zakharov and Shabat [23].

We end with a technical remark. In this introduction we have been considering u_W and τ_W as functions of the single variable x. In the body of the paper, however, it will be more convenient to think of them as functions of an infinite sequence of variables $(x, t_2, t_3, \ldots)$, or alternatively as functions of the element

$$g = \exp(xz + t_2 z^2 + t_3 z^3 + \ldots)$$

of the group Γ_+. To do this we define

$$u_W(x, t_2, t_3, \ldots) = u_{g^{-1}W}(0),$$
$$\tau_W(x, t_2, t_3, \ldots) = \tau_{g^{-1}W}(0).$$

Then $u = u_W$ will be a solution of the hierarchy (1.1) in the sense that

$$\frac{\partial}{\partial t_k} L_u = [P_u^{(k)}, L_u].$$

Note added July 1984. — We draw the reader's attention to several related papers and preprints [26-33] by Japanese authors, which we have seen since completing this work.

Summary of contents

§ 2 describes the Grassmannian of Hilbert space and its relationship with loop groups.

§ 3 describes the determinant line bundle Det on the Grassmannian, and its relationship with a central extension of the loop group. We introduce the τ-function, and calculate it explicitly for the subspaces which correspond to multisolitons.

§ 4 is an outline of the basic formal theory of the generalized KdV equations.

§ 5 describes the correspondence between points of the Grassmannian, Baker functions, and differential operators, and works out the simplest examples. We also give a characterization of the class of solutions $\mathscr{C}^{(n)}$.

§ 6 shows how Krichever's construction fits into the framework of §§ 2-5. It also contains a discussion of rings of commuting differential operators, and of the " Painlevé property " of the stationary solutions of the KdV equations.

§ 7 is devoted to the subspaces $\mathrm{Gr}_0^{(n)}$ and $\mathrm{Gr}_1^{(n)}$ of the Grassmannian which were mentioned above.

§ 8 obtains the expansion of the general τ-function as a sequence of Schur functions.

§ 9 proves that when W arises from an algebraic curve the τ-function τ_W can be expressed explicitly in terms of the θ-function of the curve.

§ 10 is an appendix explaining the connections between the theory developed in the paper and the representation theory of loop groups.

2. The Grassmannian and loop groups

In this section we shall describe the Grassmannian of Hilbert space and its relation with loop groups. The material is all fairly well known, and we shall not prove all our assertions. For a much more detailed discussion we refer the reader to [17].

The Grassmannian

Let H be a separable complex Hilbert space with a given decomposition $H = H_+ \oplus H_-$ as the direct sum of two infinite dimensional orthogonal closed subspaces. We are interested in the Grassmannian of all subspaces of H which are comparable with H_+ in the following sense.

Definition. — Gr(H) *is the set of all closed subspaces* W *of* H *such that*

(i) *the orthogonal projection* $\mathrm{pr}: W \to H_+$ *is a Fredholm operator (i.e. has finite dimensional kernel and cokernel), and*

(ii) *the orthogonal projection* $\mathrm{pr}: W \to H_-$ *is a compact operator.*

If W belongs to Gr(H), then so does the graph of any compact operator from W to $W^\perp$. Thus W lies in a subset of Gr(H) which is in one to one correspondence with the vector space $\mathscr{K}(W; W^\perp)$ of compact operators $W \to W^\perp$. This makes the Grassmannian into a Banach manifold modelled on $\mathscr{K}(H_+; H_-)$, which is given the operator-norm topology.

If $W \in \mathrm{Gr}(H)$, we shall call the index of the Fredholm operator $\mathrm{pr}: W \to H_+$ the *virtual dimension* of W (recall that the *index* of a Fredholm operator T is $\dim(\ker T) - \dim(\mathrm{coker}\, T)$). The Grassmannian is not connected: two subspaces belong to the same component if and only if they have the same virtual dimension. In the application to differential equations we shall be interested only in the component consisting of subspaces of virtual dimension zero. These subspaces are precisely the ones which are the images of embeddings $H_+ \to H$ which differ from the standard inclusion by a compact operator.

Because of the restrictions on the subspaces W belonging to Gr(H), not every invertible operator on H induces a map of Gr(H). We define the *restricted general linear group* $GL_{res}(H)$ as follows. Let us write operators $g \in GL(H)$ in the block form

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.1}$$

with respect to the decomposition $H = H_+ \oplus H_-$. Then $GL_{res}(H)$ is the closed subgroup of GL(H) consisting of operators g whose off-diagonal blocks b and c are compact operators. The blocks a and d are then automatically Fredholm. The group of connected components of $GL_{res}(H)$ is $\mathbf{Z}$, the component being determined by the index of the Fredholm operator a.

Lemma **2.2.** — *The group* $GL_{res}(H)$ *acts on* Gr(H).

Proof. — A subspace W belongs to Gr(H) precisely when it is the image of an embedding $w_+ \oplus w_- : H_+ \to H_+ \oplus H_-$ with w_+ Fredholm and w_- compact. Then its transform by the element g in (2.1) above is the image of $w'_+ \oplus w'_-$, where $w'_+ = aw_+ + bw_-$ and $w'_- = cw_+ + dw_-$. But w'_+ is Fredholm and w'_- is compact.

We can read off from the formula for w'_+ that the virtual dimension of gW differs from that of W by the index of a.

Remark. — The action of $GL_{res}(H)$ on Gr(H) is easily seen to be transitive.

We now specialize to the case where H is the space of all square-integrable complex valued functions on the unit circle $S^1 = \{z \in \mathbf{C} : |z| = 1\}$. In this space we have a natural orthonormal basis consisting of the functions $\{z^k\}$, $k \in \mathbf{Z}$. We define H_+ and H_- to be the closed subspaces spanned by the basis elements $\{z^k\}$ with $k \geq 0$ and $k < 0$, respectively. Then $H = H_+ \oplus H_-$: we shall write simply Gr for the Grassmannian corresponding to this choice of (H, H_+, H_-).

Any continuous non-vanishing function f on S^1 defines an invertible multiplication operator, again written f, on H. This induces an action on Gr in view of the following theorem.

Proposition **2.3.** — *Let* Γ *denote the group of continuous maps* $S^1 \to \mathbf{C}^\times$, *regarded as multiplication operators on* H. *Then* $\Gamma \subset GL_{res}(H)$.

If $f : S^1 \to \mathbf{C}^\times$ *is twice differentiable, then the off-diagonal blocks of the corresponding operator are of trace class (i.e. nuclear).*

Proof. — The first assertion follows from the second, as the usual topology on Γ corresponds to the norm topology on the multiplication operators, and for this a limit of operators of trace class is compact.

Now let $f = \Sigma f_k z^k$ be the Fourier expansion of f. The matrix of the corresponding operator, with respect to the basis $\{z^k\}_{k\in\mathbf{Z}}$ of H, has (i,j)-th entry f_{i-j}. We must show that the blocks $\{i \geq 0, j < 0\}$ and $\{i < 0, j \geq 0\}$ are of trace class. But a matrix (α_{ij}) is certainly of trace class if $\Sigma\,|\alpha_{ij}| < \infty$. So what we need is that

$$\sum_{i\geq 0,\, j<0} |f_{i-j}| < \infty \quad \text{and} \quad \sum_{i<0,\, j\geq 0} |f_{i-j}| < \infty,$$

that is,

$$\sum_{k>0} k\,|f_k| < \infty \quad \text{and} \quad \sum_{k>0} k\,|f_{-k}| < \infty.$$

These conditions are satisfied if f is twice differentiable, because the Fourier series of a C^1 function is absolutely convergent.

In this paper we shall be interested mainly in the action of the subgroup Γ_+ of Γ consisting of all real-analytic functions $f : S^1 \to \mathbf{C}^\times$ which extend to holomorphic functions $f : D_0 \to \mathbf{C}^\times$ in the disc $D_0 = \{z \in \mathbf{C} : |z| \leq 1\}$ satisfying $f(0) = 1$. (Here and elsewhere, when we say that a function defined on a closed set in $\mathbf{C}$ is holomorphic, we mean that it extends to a holomorphic function in a neighbourhood of the set.) We shall also consider the subgroup Γ_- of Γ consisting of functions f which extend to non-vanishing holomorphic functions in $D_\infty = \{z \in \mathbf{C} \cup \infty : |z| \geq 1\}$ satisfying $f(\infty) = 1$. Concerning this subgroup we can assert

Proposition **2.4.** — Γ_- *acts freely on* Gr.

We shall postpone the proof of this for a moment.

The stratification of Gr

We shall make much use of some special spaces $H_S \in \mathrm{Gr}$ indexed by certain subsets S of the integers: for any $S \subset \mathbf{Z}$ we define H_S to be the closed subspace of H spanned by $\{z^s\}_{s\in S}$. The kernel and cokernel of the orthogonal projection $H_S \to H_+$ are spanned by the functions $\{z^i\}$ with i belonging to $S \setminus \mathbf{N}$ and $\mathbf{N} \setminus S$, respectively; thus $H_S \in \mathrm{Gr}$ precisely when both $S \setminus \mathbf{N}$ and $\mathbf{N} \setminus S$ are finite. We denote by $\mathscr{S}$ the set of all subsets $S \subset \mathbf{Z}$ of this kind. If $S \in \mathscr{S}$, we call the number

$$\mathrm{card}(S \setminus \mathbf{N}) - \mathrm{card}(\mathbf{N} \setminus S)$$

the *virtual cardinal* of S: it is equal to the virtual dimension of H_S. A set of virtual cardinal d is simply an increasing sequence $S = \{s_0, s_1, s_2, \ldots\}$ of integers such that $s_i = i - d$ for all sufficiently large i. Let us order the set $\mathscr{S}$ by defining

$$S \leq S' \Leftrightarrow s_k \geq s'_k \quad \text{for all } k.$$

Lemma **2.5.** — *For every* $W \in \mathrm{Gr}$, *there exist sets* $S \in \mathscr{S}$ *such that* W *is the graph of a compact operator* $H_S \to H_S^\perp$, *or, equivalently, such that the orthogonal projection* $W \to H_S$ *is an isomorphism. Furthermore there is a unique minimal* S *with this property.*

We shall omit the straightforward proof of this lemma: it can be found in [17]. Let us only point out that the unique minimal S associated to W consists precisely of those integers s such that W contains an element of order s, i.e. an element of the form $\sum_{k \leq s} a_k z^k$ with $a_s \neq 0$.

A very useful corollary of the lemma is

Proposition **2.6.** — *In any* $W \in \mathrm{Gr}$, *the elements of finite order form a dense subspace, which we shall denote by* W^{alg}.

This holds because a projection $W \to H_S$ which is an isomorphism induces an isomorphism between W^{alg} and H_S^{alg}; and the elements of finite order are obviously dense in H_S.

Let us at this point return to give the proof of Proposition 2.4. Suppose that $W \in \mathrm{Gr}$, and that $g \in \Gamma_-$ is such that $gW = W$. Now g is of the form $1 + \sum_{k<0} a_k z^k$. Let $w \in W$ be an element of minimal order s_0. Then $gw - w$ is an element of W of order less than s_0. So $gw = w$, and hence $g = 1$.

The spaces $W \in \mathrm{Gr}$ which are the graphs of operators $H_S \to H_S^{\perp}$ will be called *transverse* to $H_S^{\perp}$. They form an open set U_S of Gr; lemma 2.5 asserts that the U_S cover Gr. The set of W such that S is the *minimal* indexing set with $W \in U_S$ form a closed submanifold of U_S denoted by Σ_S. Notice that W belongs to Σ_S precisely when W^{alg} has a basis $\{w_i\}_{i \geq 0}$ (in the algebraic sense) with w_i of order s_i.

The Σ_S constitute a stratification of Gr by manifolds of finite codimension. The codimension of Σ_S is the *length* of S, defined as

$$\ell(S) = \sum_{k \geq 0} (k - s_k - d),$$

where d is the virtual cardinal of S.

Scaling

For each $\lambda \in S^1$, we can consider the operator R_λ on H induced by rotating the circle S^1, that is, the operator defined by

$$R_\lambda f(z) = f(\lambda^{-1} z), \quad (f \in H).$$

If λ is a complex number with $|\lambda| \neq 1$, the operator R_λ defined by this formula is unbounded. Nevertheless, using (2.5), we can see that if $|\lambda| \leq 1$, then the operator R_λ still induces a transformation of Gr. For then the domain of R_λ includes the dense subspace H^{alg} of H consisting of functions of finite order, i.e. those whose Fourier series involve only a finite number of positive powers of z. We can therefore define $R_\lambda W$ to be the closure of the space $R_\lambda W^{\mathrm{alg}}$. To see that $R_\lambda W$ belongs to Gr, we use (2.5): if W is transverse to $H_S^{\perp}$, then clearly $R_\lambda W$ is too, and is the graph of a compact operator. We shall refer to the operators $\{R_\lambda : |\lambda| \leq 1\}$ as the semigroup of *scaling transformations* of Gr. It should be noticed that $R_\lambda W$ depends continuously both on λ and on W.

The scaling operators R_λ preserve the stratification of Gr by the Σ_S. In the sense of Morse theory, Σ_S is the stable manifold of the point H_S for the scaling flow, i.e. the set of all W such that $R_\lambda W \to H_S$ as $\lambda \to 0$.

Loop groups

We now come to the connection of the Grassmannian with loop groups. Although this will not play a very prominent role in the paper, we regard it as fundamental.

Let $H^{(n)}$ be the Hilbert space of all square integrable functions on S^1 with values in $\mathbf{C}^n$. We break up $H^{(n)}$ as $H_+^{(n)} \oplus H_-^{(n)}$, using Fourier series just as in the case $n = 1$. The group $LGL_n(\mathbf{C})$ of all continuous maps $\gamma : S^1 \to GL_n(\mathbf{C})$ acts on $H^{(n)}$ in an obvious way. Generalizing Proposition 2.3 we have

Proposition **2.7.** — $LGL_n(\mathbf{C}) \subset GL_{res}(H^{(n)})$.

The proof is exactly the same as in the case $n = 1$.

Thus $LGL_n(\mathbf{C})$ acts on $Gr(H^{(n)})$. For each $\gamma \in LGL_n(\mathbf{C})$ we set $W_\gamma = \gamma . H_+^{(n)}$. Then $zW_\gamma \subset W_\gamma$, where z denotes the operation of multiplication by the scalar-valued function z on S^1; for multiplication by z commutes with the action of γ on $H^{(n)}$, and $zH_+^{(n)} \subset H_+^{(n)}$. This leads us to introduce the

Definition. — $Gr^{(n)} = \{ W \in Gr(H^{(n)}) : zW \subset W \}$.

$Gr^{(n)}$ is a closed subspace of Gr, and $LGL_n(\mathbf{C})$ acts on it. One reason for its importance is that it is essentially the loop space ΩU_n of the unitary group U_n, i.e. the space of continuous maps $\gamma : S^1 \to U_n$ such that $\gamma(1) = 1$. To be precise, $\gamma \mapsto W_\gamma$ maps ΩU_n injectively onto a dense subspace of $Gr^{(n)}$; and indeed $Gr^{(n)}$ can be identified, if one wants, with a certain class of measurable loops in U_n.

The construction by which one associates a loop to a subspace W in $Gr^{(n)}$ is as follows. One first observes that the quotient space W/zW is n-dimensional. Let $w_1, \ldots, w_n$ be elements of W which span W/zW. Think of them as functions on the circle whose values are n-component column vectors. Then $(w_1, w_2, \ldots, w_n)$ is a function on the circle with values in $GL_n(\mathbf{C})$: call it γ. It is obvious that $\gamma . H_+^{(n)} = W$. Unfortunately the matrix entries of γ are a priori only L^2 functions, and it may not be possible to choose them continuous. If the elements $w_1, \ldots, w_n$ are chosen to be an orthonormal basis for the orthogonal complement of zW in W, then it is easy to see that the loop γ takes its values in U_n. Furthermore γ is then unique up to multiplication on the right by a constant unitary matrix.

We should notice that in the correspondence between loops and subspaces the winding number of a loop γ is minus the virtual dimension of W_γ. (This can be seen by deforming γ continuously to a standard loop with the same winding number.)

We shall now identify the Hilbert space $H^{(n)}$ with $H = H^{(1)}$ by letting the standard basis $\{ \varepsilon_i z^k : 1 \leq i \leq n, k \in \mathbf{Z} \}$ for $H^{(n)}$ correspond lexicographically to the basis $\{ z^k \}$

for H. (Here $\{\varepsilon_i\}$ denotes the standard basis for $\mathbf{C}^n$.) Thus $\varepsilon_i\, z^k$ corresponds to z^{nk+i-1}. More invariantly, given a vector valued function with components $(f_0, \ldots, f_{n-1})$, we associate to it the scalar valued function $\tilde{f}$ such that

$$\tilde{f}(z) = f_0(z^n) + zf_1(z^n) + \ldots + z^{n-1} f_{n-1}(z^n).$$

Conversely, given $\tilde{f} \in \mathrm{H}$, we have

$$f_k(z) = \frac{1}{n} \sum_{\zeta} \zeta^{-k} \tilde{f}(\zeta)$$

where ζ runs through the n-th roots of z. The isomorphism $\mathrm{H}^{(n)} \cong \mathrm{H}$ is an isometry. It makes continuous functions correspond to continuous ones, and also preserves most other reasonable classes of functions, for example: smooth, real analytic, rational, polynomial. Multiplication by z on $\mathrm{H}^{(n)}$ corresponds to multiplication by z^n on H; and $\mathrm{H}^{(n)}_+$ corresponds to H_+. From now on we shall always think of $\mathrm{Gr}^{(n)}$ as the subspace of Gr given by

$$\mathrm{Gr}^{(n)} = \{\mathrm{W} \in \mathrm{Gr} : z^n\mathrm{W} \subset \mathrm{W}\}.$$

Note that $\mathrm{Gr}^{(n)}$ is preserved by the action of the group Γ and also by the semigroup of scaling transformations.

Proposition **2.8.** — *Let* $\mathrm{W} \in \mathrm{Gr}^{(n)}$. *Then for any complex number* λ *with* $|\lambda| < 1$, *the space* $\mathrm{R}_\lambda \mathrm{W}$ *corresponds to a real analytic loop.*

The proposition implies that for the purposes of this paper we could perfectly well confine ourselves to the subspace of $\mathrm{Gr}^{(n)}$ consisting of those W that correspond to analytic loops (see (5.10) below). However, most of our arguments apply naturally to the larger space $\mathrm{Gr}^{(n)}$.

Proof of (2.8). — We have to see that there is a complementary subspace for $z^n(\mathrm{R}_\lambda \mathrm{W})$ in $\mathrm{R}_\lambda \mathrm{W}$ consisting of analytic functions. Choose a complementary subspace A for $z^n \mathrm{W}$ in W such that $\mathrm{A} \subset \mathrm{W}^{\mathrm{alg}}$ (this is possible because, as we have seen, $\mathrm{W}^{\mathrm{alg}}$ is dense in W). Then each $f \in \mathrm{A}$ has the form

$$f(z) = \sum_{-\infty}^{\mathrm{N}} c_i\, z^i$$

where the series converges for $|z| > 1$; hence the series for $f(\lambda^{-1} z)$ converges for $|z| > |\lambda|$, so that $f(\lambda^{-1} z)$ is an analytic function on S^1. Thus the space $\{f(\lambda^{-1} z) : f \in \mathrm{A}\}$ is a complement to $z^n(\mathrm{R}_\lambda \mathrm{W})$ in $\mathrm{R}_\lambda \mathrm{W}$ of the desired kind.

Rational and polynomial loops

In § 7 we shall consider two subspaces $\mathrm{Gr}^{(n)}_1$ and $\mathrm{Gr}^{(n)}_0$ of $\mathrm{Gr}^{(n)}$: they can be defined as the subspaces corresponding to rational and Laurent polynomial loops, respectively. They can also be characterized in another way, which will be more convenient for us.

Proposition **2.9**. — *The following conditions on a subspace* $W \in Gr^{(n)}$ *are equivalent.*

(i) $W = W_\gamma$ *for some rational loop* γ *(that is, a loop such that each matrix entry in* γ *is a rational function of* z *with no poles on* S^1*).*

(ii) *There exist polynomials* p *and* q *in* z *such that*

$$pH_+ \subset W \subset q^{-1} H_+.$$

(iii) W *is* commensurable *with* H_+, *i.e.* $W \cap H_+$ *is of finite codimension in both* W *and* H_+.

We denote by $Gr_1^{(n)}$ the subspace of $Gr^{(n)}$ consisting of those W that satisfy the conditions in (2.9). We define Gr_1 to be the subspace of Gr consisting of those $W \in Gr$ that satisfy condition (ii) in (2.9). Notice that we may assume that the roots of the polynomials p and q all lie in the region $\{|z| < 1\}$; for if $|c| > 1$, then $z - c$ is an invertible operator on H_+.

Example **2.10**. — For spaces $W \in Gr$ not belonging to any $Gr^{(n)}$ the condition of commensurability (2.9) (iii) does not imply condition (2.9) (ii). As an example, consider the subspace W of codimension 1 in H_+ which is the kernel of the linear map $F : H_+ \to \mathbf{C}$ defined by

$$F(f) = \text{residue}_{z=0}(e^{1/z}.f).$$

Obviously there is no polynomial p such that $pH_+ \subset W$.

Proposition **2.11**. — *The following conditions on a subspace* $W \in Gr^{(n)}$ *are equivalent.*

(i) $z^q H_+ \subset W \subset z^{-q} H_+$ *for some positive integer* q.

(ii) $W = W_\gamma$ *for some Laurent polynomial loop* γ *(by this we mean that both* γ *and* γ^{-1} *have finite Laurent expansions).*

We denote by Gr_0 the subspace of Gr consisting of those W that satisfy the condition (2.11) (i), and we set

$$Gr_0^{(n)} = Gr_0 \cap Gr^{(n)}.$$

Then Gr_0 is the union of all the $Gr_0^{(n)}$.

We note that all the Grassmannians Gr_1, Gr_0, $Gr_1^{(n)}$ and $Gr_0^{(n)}$ are invariant under the semigroup of scaling transformations, and also under the action of the group Γ_+ of holomorphic functions in the disc (defined after (2.3)). (Gr_0 and Gr_1 are preserved by Γ_+ because $gH_+ = H_+$ for any $g \in \Gamma_+$.)

It is easy to see that Gr_0 is dense in Gr. As Gr_0 is the union of a sequence of compact finite dimensional algebraic varieties (namely the Grassmannians of $z^{-q} H_+/z^q H_+$), this implies that every holomorphic function on Gr is constant.

Although it will play only a minor role in this paper, we should mention that the space Gr_0 has a cell decomposition into even-dimensional cells indexed by the same set $\mathscr{S}$ as the stratification. For $S \in \mathscr{S}$ the cell C_S consists of all $W \in \mathrm{Gr}_0$ for which W^{alg} has a basis $\{w_s\}_{s \in S}$ with w_s of the form

$$w_s = z^s + \sum_{i > s} \alpha_{si} z^i.$$

The cell C_S is homeomorphic to $\mathbf{C}^{\ell(S)}$. It is a submanifold of Gr transversal to the stratum Σ_S, which it meets in the single point H_S. On Gr_0 the scaling operators R_λ make sense for all $\lambda \in \mathbf{C}^\times$, and C_S is the "unstable manifold" of H_S for the scaling flow, i.e. the set of W such that $R_\lambda W \to H_S$ as $\lambda \to \infty$.

Finally, let us observe that H_S belongs to $\mathrm{Gr}^{(n)}$ if and only if $S + n \subset S$. For such S let us write $C_S^{(n)}$ for $C_S \cap \mathrm{Gr}^{(n)}$. The $C_S^{(n)}$ form a cell decomposition of $\mathrm{Gr}^{(n)}$, and the dimension of $C_S^{(n)}$ is $\sum_i (i - s_i - d)$, where the sum is taken only over the n integers i such that $s_i \notin S + n$, and d is the virtual cardinal of S.

3. The determinant bundle and the τ-function

In this section we are going to construct a holomorphic line bundle Det over Gr. For simplicity, we shall confine ourselves to the connected component of the Grassmannian consisting of spaces of virtual dimension zero: the symbol Gr will now denote this component. We think of Det as the "determinant bundle", that is, the bundle whose fibre over $W \in \mathrm{Gr}$ is the "top exterior power" of W. Our first task is to explain how to make sense of this.

On the Grassmannian $\mathrm{Gr}_k(\mathbf{C}^n)$ of k-dimensional subspaces of $\mathbf{C}^n$ the fibre of the determinant line bundle at $W \in \mathrm{Gr}_k(\mathbf{C}^n)$ is $\det(W) = \Lambda^k(W)$. A typical element of $\Lambda^k(W)$ can be written $\lambda w_1 \wedge w_2 \wedge \ldots \wedge w_k$, with $\lambda \in \mathbf{C}$, where $\{w_i\}$ is a basis for W. In analogy with this, an element of $\det(W)$, for $W \in \mathrm{Gr}$, will be an infinite expression $\lambda w_0 \wedge w_1 \wedge w_2 \wedge \ldots$, where $\{w_i\}$ is what we shall call an *admissible basis* for W. The crucial property of the class of admissible bases is that if $\{w_i\}$ and $\{w_i'\}$ are two admissible bases of W then the infinite matrix t relating them is of the kind that has a determinant; for we want to be able to assert that

$$\lambda w_0 \wedge w_1 \wedge w_2 \wedge \ldots = \lambda \det(t)\, w_0' \wedge w_1' \wedge w_2' \wedge \ldots$$

when $w_i = \Sigma t_{ij} w_j'$.

Let us recall (see, for example, [19]) that an operator has a determinant if and only if it differs from the identity by an operator of trace class. Now the subspaces W we are considering have the property that the projection $\mathrm{pr}: W \to H_+$ is Fredholm and of index zero. This means that W contains sequences $\{w_i\}$ such that

(i) the linear map $w: H_+ \to H$ which takes z^i to w_i is continuous and injective and has image W, and

(ii) the matrix relating $\{\mathrm{pr}(w_i)\}$ to $\{z^i\}$ differs from the identity by an operator of trace class.

Such a sequence $\{w_i\}$ will be called an *admissible basis*. (A possible choice for $\{w_i\}$ is the inverse image of the sequence $\{z^s\}_{s \in S}$ under a projection $W \to H_S$ which is an isomorphism (see (2.5).)

We shall think of $w : H_+ \to H$ as a $\mathbf{Z} \times \mathbf{N}$ matrix

$$w = \begin{pmatrix} w_+ \\ w_- \end{pmatrix}$$

whose columns are the w_i, and where $w_+ - 1$ is of trace class; the block w_- is automatically a compact operator. Then w is determined by W up to multiplication on the right by an $\mathbf{N} \times \mathbf{N}$ matrix (or operator $H_+ \to H_+$) belonging to the group $\mathscr{T}$ of all invertible matrices t such that $t - 1$ is of trace class. (The topology of $\mathscr{T}$ is defined by the trace norm.) Because operators in $\mathscr{T}$ have determinants we can define an element of Det(W) as a pair (w, λ), where $\lambda \in \mathbf{C}$ and w is an admissible basis of W, and we identify (w, λ) with (w', λ') when $w' = wt^{-1}$ and $\lambda' = \lambda \det(t)$ for some $t \in \mathscr{T}$, (We could also write (w, λ) as $\lambda w_0 \wedge w_1 \wedge \dots$)

To be quite precise, the space $\mathscr{P}$ of matrices w should be given the topology defined by the operator norm on w_- and the trace norm on $w_+ - 1$. Then $\mathscr{P}$ is a principal $\mathscr{T}$-bundle on $\mathrm{Gr} = \mathscr{P}/\mathscr{T}$, and the total space of Det is $\mathscr{P} \times_{\mathscr{T}} \mathbf{C}$ where $\mathscr{T}$ acts on $\mathbf{C}$ by $\det : \mathscr{T} \to \mathbf{C}^\times$.

Now we come to the crucial difference between the finite and infinite dimensional cases. The group $GL_n(\mathbf{C})$ acts on $\mathrm{Gr}_k(\mathbf{C}^n)$, and also on the total space of the line bundle det on it: if $g \in GL_n(\mathbf{C})$ and $w_1 \wedge \dots \wedge w_k \in \det(W)$ then $g.(w_1 \wedge \dots \wedge w_k)$ is defined as $gw_1 \wedge \dots \wedge gw_k$ in $\det(gW)$. We have seen that the corresponding group which acts on Gr is not the entire general linear group of H but the identity component of the smaller group $GL_{res}(H)$ of invertible operators in H of the form

$$(3.1) \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(with respect to the decomposition $H = H_+ \oplus H_-$), where b and c are *compact*. But this action on Gr does not automatically induce an action on Det, for if $\{w_i\}$ is an admissible basis for W then $\{gw_i\}$ is usually not an admissible basis for gW. To deal with this problem we introduce the slightly smaller group $GL_1(H)$ consisting of invertible operators g of the form (3.1), but where the blocks b and c are of trace class. The topology of $GL_1(H)$ is defined by the operator norm on a and d, and the trace norm on b and c. We shall see that the action of the identity component $GL_1(H)^0$ on Gr does lift projectively to Det. In other words there is a central extension $GL_1^\wedge$ of $GL_1(H)^0$ by $\mathbf{C}^\times$ which acts on Det, covering the action of $GL_1(H)^0$ on Gr.

To obtain a transformation of Det we must give not only a transformation g of H but also some information telling us how to replace a non-admissible basis $\{gw_i\}$ of gW by an admissible one. To do this we introduce the subgroup $\mathscr{E}$ of $GL_1(H)^0 \times GL(H_+)$ consisting of pairs (g, q) such that $aq^{-1} - 1$ is of trace class, where a is as in (3.1).

(We give $\mathscr{E}$ the topology induced by its embedding $(g, q) \mapsto (g, q, aq^{-1} - 1)$ in $\mathrm{GL}_1(\mathrm{H}) \times \mathrm{GL}(\mathrm{H}_+) \times \{\text{operators of trace class}\}$.) The definition of $\mathscr{E}$ is precisely designed to make it act naturally on the space $\mathscr{P}$ of admissible bases by

$$(g, q) . w = gwq^{-1},$$

and hence act on Det by $(g, q) . (w, \lambda) = (gwq^{-1}, \lambda)$.

The group $\mathscr{E}$ has a homomorphism $(g, q) \mapsto g$ onto $\mathrm{GL}_1(\mathrm{H})^0$. Its kernel can clearly be identified with $\mathscr{T}$. Thus we have an extension

$$\mathscr{T} \to \mathscr{E} \to \mathrm{GL}_1(\mathrm{H})^0.$$

But the subgroup $\mathscr{T}_0$ of $\mathscr{T}$ consisting of operators of determinant 1 acts trivially on Det, so that in fact the quotient group $\mathrm{GL}_1^\wedge = \mathscr{E}/\mathscr{T}_0$ acts on Det. This last group is a central extension of $\mathrm{GL}_1(\mathrm{H})^0$ by $\mathscr{T}/\mathscr{T}_0 \cong \mathbf{C}^\times$.

The extension

$$\mathbf{C}^\times \to \mathrm{GL}_1^\wedge \to \mathrm{GL}_1(\mathrm{H})^0$$

is a non-trivial fibre bundle: there is no continuous cross-section $\mathrm{GL}_1(\mathrm{H})^0 \to \mathrm{GL}_1^\wedge$, and the extension cannot be described by a continuous cocycle. But on the dense open set $\mathrm{GL}_1^{\mathrm{reg}}$ of $\mathrm{GL}_1(\mathrm{H})^0$ where a is invertible, there is a cross-section s of $\mathscr{E} \to \mathrm{GL}_1(\mathrm{H})^0$ given by $s(g) = (g, a)$; the corresponding cocycle is

$$(g_1, g_2) \mapsto \det(a_1 a_2 a_3{}^{-1}),$$

where $g_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$, and $g_3 = g_1 g_2$. We shall always make the elements of $\mathrm{GL}_1^{\mathrm{reg}}$ act on Det by means of the section s. Of course, $\mathrm{GL}_1^{\mathrm{reg}}$ is not a group, and the map s is not multiplicative. But let GL_1^+ be the subgroup of $\mathrm{GL}_1^{\mathrm{reg}}$ consisting of elements whose block decomposition has the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. Then the restriction of s to GL_1^+ is an inclusion of groups $\mathrm{GL}_1^+ \to \mathscr{E}$ and we can regard GL_1^+ as a group of automorphisms of the bundle Det. Similar remarks apply to the subgroup GL_1^-, consisting of elements of $\mathrm{GL}_1^{\mathrm{reg}}$ whose block decomposition has the form $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$. In particular the subgroups Γ_+ and Γ_- of the group of maps $\mathrm{S}^1 \to \mathbf{C}^\times$ act on Det, for $\Gamma_\pm \subset \mathrm{GL}_1^\pm$. (Cf. remarks following (2.3).)

The τ-function

We have now reached our main goal in this section, the definition of the τ-function.

Alongside the determinant bundle Det just constructed there is its dual Det*, whose fibres are the duals of the fibres of Det. A point of Det* over $\mathrm{W} \in \mathrm{Gr}$ can be taken to be a pair (w, λ), where w is an admissible basis for W, $\lambda \in \mathbf{C}$, and (w, λ) is identified with (w', λ') if $w' = wt$ and $\lambda' = \lambda \det(t)$ for some $t \in \mathscr{T}$. The action of $\mathrm{GL}_1^\wedge$

on Det induces an action on Det*. The line bundle Det* has a canonical global holomorphic section σ, defined by

$$\sigma(\mathrm{W}) = (w, \det w_+),$$

where $\mathrm{W} \in \mathrm{Gr}$, and w is an admissible basis for W. We can think of $\sigma(\mathrm{W})$ as the determinant of the orthogonal projection $\mathrm{W} \to \mathrm{H}_+$; note that $\sigma(\mathrm{W}) = 0$ if and only if W is not transverse to H_-. The section σ is not equivariant with respect to the action of Γ_+ on Det*. For each $\mathrm{W} \in \mathrm{Gr}$, the *$\tau$-function* of W is the holomorphic function $\tau_{\mathrm{W}} : \Gamma_+ \to \mathbf{C}$ defined by

$$\tau_{\mathrm{W}}(g) = \frac{\sigma(g^{-1}\mathrm{W})}{g^{-1}\delta_{\mathrm{W}}},$$

where δ_{W} is some non-zero element of the fibre of Det* over W. In general there is no canonical choice of δ_{W}, so that τ_{W} is defined only up to a constant factor. However, if W is transverse to H_-, it is natural to choose $\delta_{\mathrm{W}} = \sigma(\mathrm{W})$, so that the τ-function is given by

(3.2) $$\tau_{\mathrm{W}}(g) . g^{-1}\sigma(\mathrm{W}) = \sigma(g^{-1}\mathrm{W}) \quad (\text{for W transverse to } \mathrm{H}_-).$$

It is easy to give an explicit formula for τ_{W} as an infinite determinant.

Proposition **3.3.** — *Let $g^{-1} \in \Gamma_+$ have the block form*

$$g^{-1} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

with respect to the splitting $\mathrm{H} = \mathrm{H}_+ \oplus \mathrm{H}_-$. Then for $\mathrm{W} \in \mathrm{Gr}$, we have

(3.4) $$\tau_{\mathrm{W}}(g) = \det(w_+ + a^{-1}bw_-),$$

where w is an admissible basis of W. *In particular, if* W *is transverse to* H_- *and* τ_{W} *is normalized as in* (3.2), *then we have*

(3.5) $$\tau_{\mathrm{W}}(g) = \det(1 + a^{-1}b\mathrm{A}),$$

where $\mathrm{A} : \mathrm{H}_+ \to \mathrm{H}_-$ *is the map whose graph is* W.

The proposition follows at once from the definitions.

Example

An interesting example of a space W belonging to $\mathrm{Gr}_1^{(2)}$ is the following one, which, as we shall see, is related to the m-soliton solution of the KdV equation.

Let $p_1, \ldots, p_m$ be non-zero complex numbers such that $|p_i| < 1$ and all p_i^2 are distinct; and let $\lambda_1, \ldots, \lambda_m$ be also non-zero. Then $\mathrm{W} = \mathrm{W}_{p,\lambda}$ denotes the closure of the space of functions f which are holomorphic in the unit disc except for a pole of order $\leq m$ at the origin, and which satisfy $f(-p_i) = \lambda_i f(p_i)$ for $i = 1, \ldots, m$. To calculate τ_W we first determine the map $A: H_+ \to H_-$ whose graph is $W_{p,\lambda}$. This assigns to $f \in H_+$ the polynomial

$$A(f) = \alpha_1(f)\, z^{-1} + \ldots + \alpha_m(f)\, z^{-m}$$

such that $f + A(f)$ belongs to $W_{p,\lambda}$. Clearly each $\alpha_i(f)$ is a linear combination of $\beta_1(f), \ldots, \beta_m(f)$, where

$$\beta_i(f) = \frac{1}{2}\,(\lambda_i^{\frac{1}{2}} f(p_i) - \lambda_i^{-\frac{1}{2}} f(-p_i)),$$

for $A(f)$ is zero when the $\beta_i(f)$ vanish. In fact $\beta_i = \Sigma M_{ij}\,\alpha_j$, where

$$M_{ij} = \frac{1}{2}\,(\lambda_i^{\frac{1}{2}} - (-1)^j \lambda_i^{-\frac{1}{2}})\, p_i^{-j};$$

and $W_{p,\lambda}$ is transverse to H_- precisely when $\det(M_{ij}) \neq 0$.

To apply (3.5) we must also calculate the map $a^{-1} b : H_- \to H_+$ corresponding to the element g^{-1} of Γ_+. We write g in the form $\exp \sum_{k>0} t_k z^k$.

Suppose that $a^{-1} b$ takes z^{-k} to $f_k \in H_+$. Before determining f_k let us observe that an infinite determinant of the form

$$\det(1 + \sum_{i=1}^{m} f_i \otimes \alpha_i)$$

reduces to the determinant of the $m \times m$ matrix whose (i,j)-th entry is

$$\delta_{ij} + \alpha_i(f_j).$$

Thus $\tau_W(\mathbf{t}) = \det(M_{ij})^{-1} \det(M_{ij} + \beta_i(f_j))$.

If $\mathrm{pr}: H \to H_+$ is the projection, we find

$$\begin{aligned} f_k &= g \cdot \mathrm{pr}(g^{-1} z^{-k}) \\ &= z^{-k}\{1 - e^{\Sigma t_i z^i}(1 + c_1 z + c_2 z^2 + \ldots + c_{k-1} z^{k-1})\}, \end{aligned}$$

where $\Sigma c_i z^i$ is the expansion of $e^{-\Sigma t_j z^j}$; and so

$$M_{ij} + \beta_i(f_j) = -\beta_i\{z^{-j} e^{\Sigma t_k z^k}(1 + c_1 z + \ldots + c_{j-1} z^{j-1})\}.$$

The determinant of this matrix, after the obvious column operations have been performed on it, reduces to

$$(-1)^m \exp(\sum_{i,k} t_{2k} p_i^{2k}) \det \begin{pmatrix} p_1^{-1} \varphi_1(\theta_1 + \delta_1) & p_1^{-2} \varphi_2(\theta_1 + \delta_1) & \ldots & p_1^{-m} \varphi_m(\theta_1 + \delta_1) \\ p_2^{-1} \varphi_1(\theta_2 + \delta_2) & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ p_m^{-1} \varphi_1(\theta_m + \delta_m) & \ldots & \ldots & p_m^{-m} \varphi_m(\theta_m + \delta_m) \end{pmatrix},$$

where $\varphi_i = \cosh$ for i odd and $= \sinh$ for i even,

$$\theta_i = \sum_{k \text{ odd}} p_i^k t_k, \text{ and}$$

$$\delta_i = \frac{1}{2} \log \lambda_i.$$

The constant factor $(-1)^m \det(M_{ij})^{-1}$ in τ_W can be ignored.

In § 5 we shall see that $2\left(\dfrac{\partial}{\partial t_1}\right)^2 \log \tau_W$ is a solution to the KdV equations. It is usually called the " m-soliton " solution.

The projective multiplier on Γ_+ and Γ_-

The results of this subsection will be used only in § 9.

The actions of the groups Γ_+ and Γ_- on Gr obviously commute with each other. However, their actions on Det* do not commute, and we shall need to know the relationship between them. Note that since the discs D_0 and D_∞ are simply connected, the elements $g \in \Gamma_+$ and $\tilde{g} \in \Gamma_-$ can be written uniquely in the form $g = e^f$, $\tilde{g} = e^{\tilde{f}}$, where $f : D_0 \to \mathbf{C}$ and $\tilde{f} : D_\infty \to \mathbf{C}$ are holomorphic maps with $f(0) = \tilde{f}(\infty) = 0$. If γ is an element of either Γ_+ or Γ_-, we shall write $\mathscr{D}(\gamma)$ for the corresponding automorphism of the bundle Det*.

Proposition **3.6.** — *If $g \in \Gamma_+$ and $\tilde{g} \in \Gamma_-$, then*

$$\mathscr{D}(\tilde{g})\, \mathscr{D}(g) = c(\tilde{g}, g)\, \mathscr{D}(g)\, \mathscr{D}(\tilde{g}),$$

where, if as above $\tilde{g} = e^{\tilde{f}}$ and $g = e^f$, we have

$$c(\tilde{g}, g) = e^{S(\tilde{f}, f)}$$

and

$$S(\tilde{f}, f) = \frac{1}{2\pi i} \int_{S^1} \tilde{f}'(z) f(z)\, dz.$$

Proof. — It is immediate from the definition of the actions of $\Gamma_\pm$ on Det* that we have a formula of the kind stated, with

$$c(\tilde{g}, g) = \det(a\tilde{a}a^{-1}\tilde{a}^{-1})$$

where a and $\tilde{a}$ are the $H_+ \to H_+$ blocks of g and $\tilde{g}$. (The commutator has a determinant because, from the fact that g and $\tilde{g}$ commute, it is equal to $1 - b\tilde{c}a^{-1}\tilde{a}^{-1}$, where b and $\tilde{c}$ are the off-diagonal blocks of g and $\tilde{g}$, which are of trace class by (2.3).) The map c is a homomorphism from $\Gamma_- \times \Gamma_+$ to $\mathbf{C}$; it follows easily that it is of the desired form, with

$$S(\tilde{f}, f) = \operatorname{trace}\,[\alpha, \tilde{\alpha}],$$

where α and $\tilde{\alpha}$ are the $H_+ \to H_+$ blocks of f and $\tilde{f}$. Now, if $f = \Sigma a_i z^i$ and $\tilde{f} = \Sigma b_i z^{-i}$, the (k, k) matrix element of the commutator $[\alpha, \tilde{\alpha}]$ is

$$\sum_{m=1}^{k} a_m b_m - \sum_{m=1}^{\infty} a_m b_m.$$

The trace is therefore

$$-\sum_{m=1}^{\infty} m a_m b_m = \frac{1}{2\pi i} \int_{S^1} \tilde{f}'(z)\, f(z)\, dz.$$

as stated.

Lemma **3.7.** — *The section* σ *of* Det* *is equivariant with respect to the action of* Γ_-, *that is, we have*

$$\sigma(\tilde{g}W) = \tilde{g}\sigma(W) \quad \text{for } \tilde{g} \in \Gamma_-.$$

Lemma **3.8.** — *For* $\tilde{g} \in \Gamma_-$, *we have*

$$\tau_{\tilde{g}W}(g) = e^{S(\tilde{f},f)}\, \tau_W(g),$$

where as before $g = e^f$ *and* $\tilde{g} = e^{\tilde{f}}$.

Both lemmas follow at once from the definitions.

General remarks

In the theory of loop groups like the group L of smooth maps $S^1 \to GL_n(\mathbf{C})$ the existence of a certain central extension

$$\mathbf{C}^\times \to \hat{L} \to L$$

plays an important role. This extension (at least over the identity component of L) is the restriction of the central extension $GL_1^\wedge$ constructed in this section, when L is embedded in the usual way in $GL_1(H)$.

On the level of Lie algebras the extension can be described very simply for the loop group LG of any reductive group G. The Lie algebra of LG is the vector space $L\mathfrak{g}$ of loops in the Lie algebra $\mathfrak{g}$ of G, and the extension is defined by the cocycle

$$\beta : L\mathfrak{g} \times L\mathfrak{g} \to \mathbf{C}$$

given by
$$\beta(f_1, f_2) = \frac{1}{2\pi}\int_0^{2\pi} \langle f_1'(\theta), f_2(\theta)\rangle\, d\theta,$$

where $\langle\ ,\ \rangle$ is a suitably normalized invariant bilinear form on $\mathfrak{g}$.

The existence of the corresponding extension of groups is less obvious (cf. [18]), partly because it is topologically non-trivial as a fibre bundle. The discussion in this section provides a concrete realization of $\hat{L}$ as a group of holomorphic automorphisms of the line bundle Det, in the case $G = GL_n(\mathbf{C})$. For the elements of $\hat{L}$ above $\gamma \in L$ are precisely the holomorphic bundle maps $\hat{\gamma} : \mathrm{Det} \to \mathrm{Det}$ which cover the action of γ on Gr. (For given γ the possible choices of $\hat{\gamma}$ differ only by multiplication by constants, as any holomorphic function on Gr is constant. (Cf. remark following (2.11)).)

The corresponding central extension of the loop group of any complex reductive group (characterized by its Lie algebra cocycle) can be constructed in a similar way as a group of holomorphic automorphisms of a complex line bundle, and conversely the holomorphic line bundle is determined by the group extension. This is explained in [17]. But in the general case the line bundle does not have such a simple description.

4. Generalized KdV equations and the formal Baker function

The *n-th generalized KdV hierarchy* consists of all evolution equations for $n-1$ unknown functions $u_0(x,t), \ldots, u_{n-2}(x,t)$ that can be written in the form $\partial \mathrm{L}/\partial t = [\mathrm{P}, \mathrm{L}]$, where L is the n-th order ordinary differential operator

$$\mathrm{L} = \mathrm{D}^n + u_{n-2}\,\mathrm{D}^{n-2} + \ldots + u_1\,\mathrm{D} + u_0$$

and P is another differential operator. (As usual, D denotes $\partial/\partial x$.) The possible operators P are essentially determined by the requirement that $[\mathrm{P}, \mathrm{L}]$ should have order (at most) $n-2$. A very simple description of them is available if we work in the algebra of *formal pseudo-differential operators*, which we denote by Psd.

A formal pseudo-differential operator is, by definition, a formal series of the form

$$\mathrm{R} = \sum_{-\infty}^{\mathrm{N}} r_i(x)\,\mathrm{D}^i$$

for some $\mathrm{N} \in \mathbf{Z}$. The coefficients $r_i(x)$ are supposed to lie in some algebra of smooth functions of x. To multiply two such operators, we need to know how to move D^{-1} across a function $r(x)$: the rule for this,

$$\mathrm{D}^{-1}\,r = \sum_{j=0}^{\infty} (-1)^j\, r^{(j)}\,\mathrm{D}^{-1-j},$$

follows easily from the basic rule

$$\mathrm{D}r = r\,\mathrm{D} + \partial r/\partial x \tag{4.1}$$

determining the composition of *differential* operators. It is easy to check that this makes Psd into an associative algebra.

Proposition **4.2**. — *In the algebra* Psd, *the operator* L *has a unique n-th root of the form*

$$\mathrm{L}^{1/n} = \mathrm{Q} = \mathrm{D} + \sum_{1}^{\infty} q_i\,\mathrm{D}^{-i}.$$

The coefficients q_i are certain universal differential polynomials in the u_i; if we assign to $u_i^{(j)}$ the weight $n-i+j$, then q_i is homogeneous of weight $i+1$.

Proof. — Equating coefficients of powers of D in the equality $\mathrm{Q}^n = \mathrm{L}$, we find that

$$u_{n-i-1} = nq_i + \alpha_i,$$

where α_i is some differential polynomial in $q_1, \ldots, q_{i-1}$ (here we have set $u_j = 0$ if $j < 0$). We claim that if we give $q_i^{(j)}$ weight $i+j+1$ then α_i is homogeneous of weight $i+1$. Granting that, it is clear that the above equations can be solved uniquely for the q_i, and that these have the form stated.

The homogeneity of the α_i is most easily seen as follows. Consider the algebra of formal pseudo-differential operators whose coefficients are differential polynomials in the q_i (which we think of for the moment as abstract symbols, rather than as fixed functions of x). Call such an operator homogeneous of weight r if the coefficient of D^i is homogeneous of weight $r - i$ (thus D has weight 1). From the homogeneity of the basic rule (4.1) it follows at once that the product of two operators that are homogeneous of weights r and s is homogeneous of weight $r + s$. Since Q is homogeneous of weight 1, Q^n must be homogeneous of weight n.

If $R = \Sigma r_i D^i$ is a formal pseudo-differential operator, we shall write R_+ for the " differential operator part " $R_+ = \sum_{i \geq 0} r_i D^i$, and $R_- = \sum_{i<0} r_i D^i$. Thus $R = R_+ + R_-$.

Proposition **4.3.** — *The equation*

$$\partial L/\partial t = [L_+^{r/n}, L] \tag{4.4}$$

is equivalent to a system of evolution equations

$$\frac{\partial u_i}{\partial t} = f_i$$

for the coefficients $u_0, \ldots, u_{n-2}$ *of* L. *The* f_i *are differential polynomials in the* u_j, *homogeneous of weight* $n + r - i$.

Proof. — Note first that $L_+^{r/n}$ denotes $(L^{r/n})_+$; $L^{r/n}$ is defined as Q^r. The only part of the proposition that is not obvious from what precedes is that the commutator in (4.4) is an operator of order at most $n - 2$. But that follows at once from the equality

$$[L_+^{r/n}, L] = [- L_-^{r/n}, L].$$

(Of course $L^{r/n}$ and L commute, because they are both powers of $Q = L^{1/n}$.)

The equation (4.4) is called the *r-th equation of the n-th KdV hierarchy*. It is trivial if r is a multiple of n, because then $L_+^{r/n} = L^{r/n}$ is just an integral power of L.

It is usual to think of the equations (4.4) as defining flows on some space of functions $\{u_0(x), \ldots, u_{n-2}(x)\}$: it is then a basic fact that the flows corresponding to different values of r commute. For this assertion to make sense, we need to identify some class of functions on which the flows can be proved to exist, that is, we need to prove existence and uniqueness theorems for solutions of the equations (4.4). However, the analytic problems involved here are in a sense irrelevant: the basic " infinitesimal " fact underlying the commutativity can be formulated in a purely algebraic way. We refer to [22] for a very simple proof of this algebraic version of the commutativity. In the present paper none of these questions need concern us, because for the special class of solutions that we are interested in, both the existence of the flows and their commutativity will be clear from the construction.

The formal Baker function

The main idea in all studies of solutions of the equations (4.4) is this: as L changes in time, we try to follow the evolution of the eigenfunctions of L by comparing them with the eigenfunctions of the constant operator D^n. To do that, we find an operator K such that $K^{-1}LK = D^n$; then if ψ_0 is an eigenfunction of D^n, $\psi = K\psi_0$ will be an eigenfunction of L. The algebra Psd enables us to give one realization of this idea.

Proposition **4.5.** — *There is an operator* $K \in \mathrm{Psd}$ *of the form*

$$K = 1 + \sum_{1}^{\infty} a_i(x)\, D^{-i} \tag{4.6}$$

such that $K^{-1}LK = D^n$. *Such a* K *is unique up to right multiplication by a constant coefficient operator of the form* $1 + c_1 D^{-1} + \dots$.

Proof. — Only constant coefficient operators commute with D^n, so the statement about uniqueness is trivial. To prove existence, we simply compare coefficients of powers of D in the equality $LK = KD^n$; this gives equations $\partial a_i/\partial x = \dots$, where the right hand side involves only a_j with $j < i$; we can therefore solve these equations successively to get suitable a_i.

Proposition 4.5 can be reformulated as follows.

Proposition **4.7.** — *The equation* $L\psi = z^n\psi$ *has a solution in the space of formal series of the form*

$$\psi = e^{xz}\left(1 + \sum_{1}^{\infty} a_i(x)\, z^{-i}\right). \tag{4.8}$$

The solution ψ *is unique up to multiplication by a series with constant coefficients of the form* $1 + c_1 z^{-1} + \dots$.

The series ψ in (4.8) is called the *formal Baker function* of L. The solutions of the KdV equations that we are going to construct are characterized by the property that this formal series actually converges (for $|z|$ sufficiently large). As we mentioned in the introduction, among these solutions are the rank 1 algebro-geometric solutions of Krichever: it was essentially in that context that the function ψ was originally introduced by Baker [3].

The intuitive reason for the equivalence of (4.5) and (4.7) was explained above: since $K^{-1}LK = D^n$, we expect the solutions of the equation $L\psi = z^n\psi$ to be of the form $\psi = K\psi_0$, where ψ_0 is a solution of $D^n\psi_0 = z^n\psi_0$. If we take $\psi_0 = e^{xz}$, then formally it is clear that $\psi = K\psi_0$ should be given by (4.8). We can make this argument rigorous as follows. Let M be the space of all formal expressions $f = e^{xz}\tilde{f}$, where $\tilde{f}$ is a formal series

$$\tilde{f} = \sum_{-\infty}^{N} f_i(x)\, z^i \quad \text{(for some N)}.$$

Differential operators act on M in an obvious way: the action of D on M is given by

$$De^{xz}\tilde{f} = e^{xz}(D+z)\tilde{f}.$$

If we let D^{-1} act on M by

$$D^{-1}e^{xz}\tilde{f} = e^{xz}(D+z)^{-1}\tilde{f},$$

where $(D+z)^{-1}$ is interpreted as the formal series $z^{-1} - Dz^{-2} + \ldots$, it is easy to check that this makes M into a module over the algebra Psd. If $R = \Sigma r_i(x)\, D^i \in \mathrm{Psd}$, then

$$Re^{xz} = e^{xz}(\Sigma r_i(x)\, z^{-i}),$$

so that M is in fact a free Psd-module of rank 1, with generator $e^{xz} \in M$.

The KP equations

It will often be convenient for us to regard the n-th KdV hierarchy (for any n) as embedded in a certain " universal KdV hierarchy " of evolution equations in infinitely many variables; for brevity we shall follow [5] and call these equations the KP (for Kadomtsev-Petviashvili) hierarchy. The KP equations are defined as follows. Let Q be a general first order formal pseudo-differential operator of the form

$$Q = D + \sum_1^{\infty} q_i(x)\, D^{-i}$$

(in general, such a Q will not be the n-th root of a differential operator for any n).

Proposition **4.10.** — *The equation*

$$\frac{\partial Q}{\partial t} = [Q^r_+, Q] \tag{4.11}$$

is equivalent to a system of evolution equations

$$\frac{\partial q_i}{\partial t} = f_i$$

for the (infinitely many) functions $q_i(x, t)$, $i \geq 1$. The f_i are certain universal differential polynomials in the q_i, homogeneous of weight $r + i + 1$ if we give $q_i^{(j)}$ weight $i + j + 1$.

The proof is the same as that of (4.3). We call (4.11) the *r-th equation of the KP hierarchy.*

Proposition **4.12.** — *The assignment $L \mapsto L^{1/n} = Q$ sets up a $1 - 1$ correspondence between solutions of the n-th KdV hierarchy and solutions Q of the KP hierarchy such that Q^n is a differential operator.*

Proof. — It is trivial that if Q satisfies (4.11) then $L = Q^n$ satisfies (4.4). We refer to [22] for the proof of the converse, which is only slightly harder.

The scaling transformation

Proposition **4.13.** — *Let* $Q = D + \Sigma q_i D^{-i}$ *be any solution of the r-th equation* (4.11). *For any non-zero complex number* λ, *let* $R_\lambda Q = D + \Sigma q_i^{(\lambda)} D^{-i}$, *where the coefficients* $q_i^{(\lambda)}$ *are defined by*

$$q_i^{(\lambda)}(x, t) = \lambda^{i+1} q_i(\lambda x, \lambda^r t).$$

Then $R_\lambda Q$ *is another solution of* (4.11).

Proof. — This follows trivially from the assertion in (4.10) about the homogeneity of the f_i.

We call the operation R_λ the *scaling transformation* of the solutions to the KP equations. Notice that each variable gets rescaled by the power of λ corresponding to its weight. The scaling transformations clearly act on the solutions to the n-th KdV hierarchy (for any n).

Note on the literature. — Our construction of the KdV equations follows closely the exposition in [14]. The basic idea of using fractional powers of L first appeared in the 1976 paper of Gel'fand and Dikii [9], and has been used extensively in the literature since then. In [5] this idea is attributed to Sato (1981).

5. The Baker function

In this section Gr and $\mathrm{Gr}^{(n)}$ will denote the component of the Grassmannians consisting of spaces of virtual dimension zero. We are going to associate to each $W \in \mathrm{Gr}$ a "Baker function" ψ_W, and also a sequence of differential operators defined in terms of ψ_W.

We recall from § 2 that the group Γ_+ of holomorphic maps $g : D_0 \to \mathbf{C}^\times$ with $g(0) = 1$ acts on Gr. Given a space $W \in \mathrm{Gr}$, we set

$$\Gamma_+^W = \{g \in \Gamma_+ : g^{-1}W \text{ is transverse to } H_-\}.$$

From now on we shall refer to spaces transverse to H_- simply as *transverse*. From § 3 it follows that Γ_+^W is the complement of the zero set of the τ-function $\tau_W : \Gamma_+ \to \mathbf{C}$; in particular it is a dense open subset of Γ_+. (We admit for the moment the fact that Γ_+^W is not empty, that is, that the holomorphic function τ_W is not identically zero: this will be proved in § 8.)

Proposition **5.1.** — *For each* $W \in \mathrm{Gr}$ *there is a unique function* $\psi_W(g, z)$, *defined for* $g \in \Gamma_+^W$ *and* $z \in S^1$, *such that*

(i) $\psi_W(g, \cdot) \in W$ *for each fixed* $g \in \Gamma_+^W$

(ii) ψ_W *has the form*

$$\psi_W = g(z)\,(1 + \sum_1^\infty a_i(g)\, z^{-i}). \tag{5.2}$$

The coefficients a_i *are analytic functions on* Γ_+^W; *they extend to meromorphic functions on the whole of* Γ_+.

The proof of the last sentence depends on the properties of the τ-function, and will be given later in this section. The rest of the proposition is trivial: the infinite series in (5.2) is simply the unique function of that form that lies in the transverse space $g^{-1}W$, that is, it is the inverse image of 1 under the orthogonal projection $g^{-1}W \to H_+$.

We call ψ_W the *Baker function* of W.

Now, each $g \in \Gamma_+$ can be written uniquely in the form

$$(5.3) \qquad g(z) = \exp(xz + t_2 z^2 + t_3 z^3 + \dots)$$

with $x, t_i \in \mathbf{C}$. When g is written in this way, we shall write $\psi_W(x, \mathbf{t}, z)$ instead of $\psi_W(g, z)$. Here $\mathbf{t}$ stands for $(t_2, t_3, \dots)$. In this notation, ψ_W is a "function of infinitely many variables" of the form

$$(5.4) \qquad \psi_W(x, \mathbf{t}, z) = \exp(xz + t_2 z^2 + \dots)(1 + \sum_1^\infty a_i(x, \mathbf{t})\, z^{-i}).$$

Proposition 5.5. — *For each integer* $r \geq 2$, *there is a unique differential operator* P_r *of the form*

$$P_r = D^r + p_{r2} D^{r-2} + \dots + p_{r,r-1} D + p_{rr}$$

such that

$$(5.6) \qquad \frac{\partial \psi_W}{\partial t_r} = P_r \psi_W.$$

(Here as usual $D = \partial/\partial x$.*) The coefficients* p_{ri} *are certain universal differential polynomials in the functions* a_i *in* (5.4).

Proof. — From (5.4), we have

$$\frac{\partial \psi_W}{\partial t_r} = g(z)(z^r + a_1 z^{r-1} + O(z^{r-2})).$$

On the other hand, $D^r \psi_W$ also has this form, and in general we have

$$D^q \psi_W = g(z)(z^q + O(z^{q-1})).$$

Comparing coefficients, we see at once that there is a unique operator P_r of the form stated such that

$$(5.7) \qquad \frac{\partial \psi_W}{\partial t_r} - P_r \psi_W = g(z)(O(z^{-1})).$$

Now, since ψ_W lies in W for each fixed $(x, \mathbf{t})$, the same is true of the derivatives $\partial\psi_W/\partial t_r$ and $D^q \psi_W$. Hence the left hand side of (5.7) lies in W for each fixed value of $(x, \mathbf{t})$ for which it is defined, that is, for which the corresponding g belongs to Γ_+^W. But the right hand side of (5.7) belongs to gH_-. As $g^{-1}W$ is transverse, both sides must vanish.

Now let

$$K = 1 + \sum_1^\infty a_i(x, \mathbf{t})\, D^{-i}$$

be the formal integral operator corresponding to ψ_W (see § 4). Equation (5.6) can be written in the form

$$\textbf{(5.8)} \qquad \frac{\partial K}{\partial t_r} + K\,D^r = P_r\,K,$$

so that in particular we have

$$P_r = (K\,D^r\,K^{-1})_+ = Q^r_+,$$

where we have set $Q = KDK^{-1}$. Thus Q is a formal pseudo-differential operator of the form

$$Q = D + \sum_1^\infty q_i(x, \mathbf{t})\,D^{-i}.$$

Proposition **5.9.** — *The coefficients q_i of Q satisfy the equations of the KP hierarchy; that is, we have*

$$\frac{\partial Q}{\partial t_r} = [Q^r_+, Q].$$

Each q_i is a meromorphic function of all the variables $(x, \mathbf{t})$.

Proof. — Differentiating the relation defining Q and rewriting, we find

$$\frac{\partial Q}{\partial t_r} = [(\partial K/\partial t_r)\,K^{-1}, Q].$$

On the other hand, from (5.8) we have

$$\frac{\partial K}{\partial t_r}K^{-1} = P_r - K\,D^r\,K^{-1} = Q^r_+ - Q^r,$$

so the proposition follows at once.

Recall from § 2 that rescaling z induces an action $W \mapsto R_\lambda W$ of the semigroup of complex numbers λ with $|\lambda| \leq 1$ on Gr.

Proposition **5.10.** — *The Baker function corresponding to the space $R_\lambda W$ is given by*

$$\psi_{R_\lambda W}(x, t_2, t_3, \ldots; z) = \psi_W(\lambda x, \lambda^2 t_2, \lambda^2 t_3, \ldots; \lambda^{-1} z).$$

If Q is the solution of the KP equations corresponding to W, then the solution corresponding to $R_\lambda W$ is $R_\lambda Q$ (see (4.13)).

The proof is trivial.

We now specialize to the case $W \in Gr^{(n)}$.

Proposition **5.11.** — *If $W \in Gr^{(n)}$, then*

$$P_n\,\psi_W = z^n\,\psi_W.$$

Moreover, the functions a_i, and hence also all the operators P_r, are independent of $t_n, t_{2n}, t_{3n}, \ldots$.

Proof. — From (5.4) and (5.6) we see that

$$\mathrm{P}_n \psi_{\mathrm{W}} - z^n \psi_{\mathrm{W}} = g(z) \sum_1^\infty \frac{\partial a_i}{\partial t_n} z^{-i}.$$

For $\mathrm{W} \in \mathrm{Gr}^{(n)}$, the left hand side of this expression lies in W for each fixed $(x, \mathbf{t})$; it therefore vanishes by the same argument as in the proof of (5.5). That proves the first statement in the proposition, and also that the a_i are independent of t_n. Since obviously $\mathrm{Gr}^{(n)} \subset \mathrm{Gr}^{(rn)}$ for all $r \geq 1$, the a_i are independent of t_{rn} too.

Since the a_i are independent of t_n, the operator K is also, so (5.8) gives

$$\mathrm{P}_n = \mathrm{K}\,\mathrm{D}^n\,\mathrm{K}^{-1} = \mathrm{Q}^n.$$

Thus if $\mathrm{W} \in \mathrm{Gr}^{(n)}$ then Q^n is a *differential* operator. Write L for $\mathrm{P}_n = \mathrm{Q}^n$; then L has the form $\mathrm{D}^n + u_{n-2}\,\mathrm{D}^{n-2} + \ldots + u_0$. Combining (5.9) and (4.12), we get the main result of this section.

Corollary **5.12.** — *If* $\mathrm{W} \in \mathrm{Gr}^{(n)}$, *the coefficients of the operator* L *satisfy the equations of the n-th KdV hierarchy, that is, we have*

$$\frac{\partial \mathrm{L}}{\partial t_r} = [\mathrm{L}_+^{r/n}, \mathrm{L}].$$

Let us reformulate this slightly. For each $\mathrm{W} \in \mathrm{Gr}^{(n)}$, let L_{W} denote the operator L evaluated for $t_2 = t_3 = \ldots = 0$. The coefficients $u_0, \ldots, u_{n-2}$ of L_{W} are functions of one variable x: they are the " initial values " of the KdV flows. Let $\mathscr{C}^{(n)}$ be the space of all L_{W} for $\mathrm{W} \in \mathrm{Gr}^{(n)}$. The map $\mathrm{Gr}^{(n)} \to \mathscr{C}^{(n)}$ is not one to one: however, from (4.7) we see that $\mathrm{L}_{\mathrm{W}} = \mathrm{L}_{\mathrm{W}'}$ precisely when $\mathrm{W} = \gamma \mathrm{W}'$, where γ is a function of the form $1 + c_1 z^{-1} + \ldots$ Since multiplication by γ commutes with the action of Γ_+, we can restate (5.12) as follows.

Proposition **5.13.** — *The action of* Γ_+ *on* $\mathrm{Gr}^{(n)}$ *induces an action on the space* $\mathscr{C}^{(n)}$. *The flow* $\mathrm{W} \mapsto \exp(t_r z^r)\,\mathrm{W}$ *on* $\mathrm{Gr}^{(n)}$ *induces the r-th KdV flow on* $\mathscr{C}^{(n)}$.

Since Γ_+ is commutative, it is obvious that the different KdV flows on $\mathscr{C}^{(n)}$ commute.

Examples

To obtain the simplest interesting example of a space in $\mathrm{Gr}^{(2)}$ we choose $p \in \mathbf{C}$ so that $0 < |p| < 1$, and $\lambda \in \mathbf{C}^\times$, and define $\mathrm{W}_{p,\lambda}$ as the L^2 closure of the space of functions f which are holomorphic in $|z| \leq 1$ except for a possible simple pole at the origin, and which satisfy $f(-p) = \lambda f(p)$.

The Baker function of $\mathrm{W}_{p,\lambda}$ must be of the form

$$\psi(\mathbf{t}, z) = e^{\Sigma t_k z^k}(1 + a(\mathbf{t})/z).$$

(We write here $\mathbf{t} = (t_1, t_2, \ldots)$, where t_1 is identified with x.) From the condition $\psi(\mathbf{t}, -p) = \lambda\psi(\mathbf{t}, p)$ we find

$$a(\mathbf{t}) = -p \tanh(\theta + \alpha),$$

where $\theta = \sum_{k \text{ odd}} p^k t_k$, and $e^{2\alpha} = \lambda$.

The second-order differential operator L such that $L\psi = z^2 \psi$ is $D^2 - 2a'$, i.e.

$$D^2 + 2p^2 \operatorname{sech}^2(\theta + \alpha).$$

This is the well known one-soliton solution of the KdV equation.

More generally, we have the subspace $W_{\mathbf{p},\lambda}$ introduced in § 3 which depends on m points $p_1, \ldots, p_m$ of the disc $|z| < 1$ and m parameters $\lambda_1, \ldots, \lambda_m \in \mathbf{C}^\times$. The corresponding solution of the KdV equation is called the *m-soliton* solution. We shall give an expression for it below in terms of the τ-function which was calculated in § 3; but let us at present notice the obvious fact that it depends on t only through $e^{\theta_i + \alpha_i}$, where $\theta_i = \sum_{k \text{ odd}} p_i^k t_k$ and $e^{2\alpha_i} = \lambda_i$. This is because the orbit of $W_{\mathbf{p},\lambda}$ under Γ_+ is isomorphic to $(\mathbf{C}^\times)^m$: in fact if $\gamma : D_0 \to \mathbf{C}^\times$ is an element of Γ_+ then $\gamma . W_{\mathbf{p},\lambda} = W_{\mathbf{p},\mu}$, where $\mu_i = \gamma(p_i)\, \gamma(-p_i)^{-1} \lambda_i$.

The Baker function and the τ-function

We now turn to the proof of the last part of (5.1), concerning the properties of the functions a_i. It depends on the formula (5.14) below, relating the Baker function to the τ-function, which we mentioned in the Introduction as central to the theory. We return to the case of an arbitrary space $W \in \mathrm{Gr}$ (not necessarily in any $\mathrm{Gr}^{(n)}$). Let us write

$$\widetilde{\psi}_W(g, z) = 1 + \sum_1^\infty a_i(g)\, z^{-1}$$

for the infinite series in (5.2). Clearly $\widetilde{\psi}_W$ extends to an analytic function of z in the region $|z| > 1$ (for each fixed $g \in \Gamma_+^W$). For $\zeta \in \mathbf{C}$ we write q_ζ for the map

$$q_\zeta(z) = 1 - z/\zeta.$$

Obviously

$$q_\zeta \in \Gamma_+ \quad \text{for } |\zeta| > 1.$$

Proposition **5.14.** — *For* $g \in \Gamma_+^W$ *and* $|\zeta| > 1$ *we have*

$$\widetilde{\psi}_W(g, \zeta) = \tau_W(g . q_\zeta)/\tau_W(g).$$

Proof. — It follows easily from (3.2) that the right hand side is equal to $\tau_{g^{-1}W}(q_\zeta)$. The left hand side is characterized as the unique function of the form $1 + \Sigma a_i \zeta^{-i}$ whose boundary value as $|\zeta| \to 1$ lies in the transverse space $g^{-1}W$. Hence the proposition follows at once if we apply the next lemma to $g^{-1}W$.

Lemma **5.15.** — *Let* $W \in \mathrm{Gr}$ *be transverse, and let* f_0 *be the unique element of* H_- *such that* $1 + f_0 \in W$. *Then for* $|\zeta| > 1$, *we have*

$$\tau_W(q_\zeta) = 1 + f_0(\zeta).$$

Proof. — We use the formula (3.5). When q_ζ^{-1} is written in the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, the map $b : H_- \to H_+$ takes z^{-k} to $\zeta^{-k} q_\zeta^{-1}$; thus $a^{-1} b$ is the map of rank one that takes $f \in H_-$ to the constant function $f(\zeta)$. The map $a^{-1} b A$ is thus also of rank one, and the infinite determinant

$$\tau_W(q_\zeta) = \det(1 + a^{-1} b A)$$

is equal to

$$1 + \operatorname{trace}(a^{-1} b A).$$

Since A maps 1 to $f_0(z)$, the lemma follows.

If we write g in the form (5.3), and correspondingly write q_ζ in the form

$$q_\zeta(z) = \exp \log(1 - z/\zeta) = \exp(-\sum_1^\infty z^k/k\zeta^k),$$

then (5.14) takes the form

$$\tilde{\psi}_W(x, \mathbf{t}, \zeta) = \tau_W(x - 1/\zeta, t_2 - 1/2\zeta^2, \ldots)/\tau_W(x, \mathbf{t}). \tag{5.16}$$

The fact that the functions $a_i(x, \mathbf{t})$ are meromorphic follows at once from this formula: indeed, if we expand the numerator in a Taylor series, we see that each a_i has the form

$$a_i = P_i \tau/\tau$$

where P_i is a polynomial differential operator in $\partial/\partial x$, $\partial/\partial t_2, \ldots, \partial/\partial t_i$. For example, we have

$$a_1 = -(\partial\tau/\partial x)/\tau$$

$$a_2 = \frac{1}{2}(\partial^2 \tau/\partial x^2 - \partial\tau/\partial t_2)/\tau.$$

The proof of (5.1) is now complete.

We can be more precise about the orders of the poles of the functions a_i. Let us fix the values of the variables $t_2, t_3, \ldots$, say $t_k = t_k^0$, and regard a_i as a meromorphic function of one variable x.

Proposition **5.17.** — *The poles of the function* $a_i(x, \mathbf{t}^0)$ *have order at most* i.

In the case of a_1, this follows at once from the formula above and the fact that τ is analytic. For $i > 1$, however, that is not so; for example, if we had $\tau = x^n + t_2$, then the corresponding function $a_2(x, \mathbf{0})$ would have a pole of order n at the origin. Our proof of (5.17) uses the expansion of the τ-function in terms of Schur functions: it will be given in § 8.

Corollary **5.18**. — *For* $W \in \mathrm{Gr}^{(n)}$, *the differential operators* $L_W \in \mathscr{C}^{(n)}$ *have only regular singular points (except for the point at infinity); that is, the coefficient* u_i *of* D^i *has poles of order at most* $n - i$.

Proof. — Recall that $L_W = K D^n K^{-1}$, where $K = 1 + \Sigma a_i(x) D^{-i}$. Thus if we give $a_k^{(j)}$ weight $k + j$, then u_i is a homogeneous differential polynomial in the a_k of weight $n - i$ (cf. the proof of (4.2)). Thus the corollary follows at once from (5.17).

Finally, we note that the coefficients u_i of L can be expressed directly in terms of the τ-function. In the case $n = 2$, L has the form $D^2 + u_W$, where

$$u_W = -2\frac{\partial a_1}{\partial x} = 2\frac{\partial^2}{\partial x^2}\log \tau_W. \tag{5.19}$$

However for $n > 2$ the explicit formulae become very complicated.

The class $\mathscr{C}^{(n)}$

We have shown how to associate an n-th order differential operator

$$L_W = D^n + u_{n-2}(x) D^{n-2} + \ldots + u_0(x), \tag{5.20}$$

with meromorphic coefficients and only regular singular points, to a space $W \in \mathrm{Gr}^{(n)}$. We shall now describe the inverse process of associating a space W to a differential operator L. This cannot be done for an arbitrary operator, even one which is meromorphic with regular singular points. We do not know an altogether satisfying description of the desired class $\mathscr{C}^{(n)}$; roughly speaking, it consists of the operators whose formal Baker functions converge for large z.

Suppose that L is of the form (5.20), with coefficients defined and smooth in an open interval I containing the origin. The formal Baker function

$$\psi(x, z) = e^{xz}\{1 + a_1(x) z^{-1} + a_2(x) z^{-2} + \ldots\}$$

of L was introduced in § 4. It is a formal series whose coefficients a_i are smooth functions defined in the interval I, and it is uniquely determined by L if we normalize it so that $\psi(0, z) = 1$. If the n formal series

$$\psi(0, z), D\psi(0, z), \ldots, D^{n-1}\psi(0, z) \tag{5.21}$$

(which belong to the field $\mathbf{C}((z^{-1}))$) converge for large z, then by a scaling transformation we can make them converge for $|z| > 1 - \varepsilon$, so that they define n elements $\psi_0, \psi_1, \ldots, \psi_{n-1}$ of our Hilbert space H. We should like to define the corresponding $W \in \mathrm{Gr}^{(n)}$ as the closed z^n-invariant subspace of H generated by $\psi_0, \ldots, \psi_{n-1}$, i.e. as γH_+, where γ is the $(n \times n)$-matrix-valued function $(\psi_0, \psi_1, \ldots, \psi_{n-1})$ on the circle. (In regarding γ as a matrix-valued function we are using the identification $H \cong H^{(n)}$ described in § 2.) For this to be possible we need to know that γ is a loop of winding number zero in $GL_n(\mathbf{C})$ — otherwise W^{alg} would turn out to be bigger than the space spanned algebraically by $\{z^{nk}\psi_i\}_{k\geq 0,\, 0\leq i<n}$. Making γ explicit according to the formulae of § 2, we find

$$\gamma(z)=\begin{pmatrix} 1 & \zeta_1 & \dots & \zeta_1^{n-1} \\ 1 & \zeta_2 & \dots & \zeta_2^{n-1} \\ \vdots & & & \\ 1 & \zeta_n & \dots & \zeta_n^{n-1}\end{pmatrix}^{-1}\begin{pmatrix}\psi_0(\zeta_1) & \psi_1(\zeta_1) & \dots & \psi_{n-1}(\zeta_1) \\ \psi_0(\zeta_2) & \psi_1(\zeta_2) & \dots & \psi_{n-1}(\zeta_2) \\ \vdots & & & \\ \psi_0(\zeta_n) & \psi_1(\zeta_n) & \dots & \psi_{n-1}(\zeta_n)\end{pmatrix}$$

where $\zeta_1, \dots, \zeta_n$ are the n-th roots of z. But $\psi_k(z) \sim z^k$ as $z \to \infty$. So γ is holomorphic in $|z| > 1 - \varepsilon$, and $\gamma(z) \to 1$ as $z \to \infty$. By a further rescaling, if necessary, we can therefore ensure that $\gamma(z)$ is invertible for $|z| > 1 - \varepsilon$, as we want.

Let us notice that the series $D^k\psi(0, z)$ depend only on the jets (i.e. Taylor series) at the origin of the coefficients u_i of L, and that conversely the series $D^k\psi(0, z)$ determine the jets of the u_i at 0. The space W which we have just constructed has its own Baker function ψ_W, which in turn defines a differential operator L_W with coefficients meromorphic in the entire complex plane. (For brevity, we shall write $\psi_W(x, z)$ for the Baker function evaluated at $t_2 = t_3 = \dots = 0$.) Because both $D^k\psi(0, z)$ and $D^k\psi_W(0, z)$ belong to W and are of the form z^k + (lower terms), it follows by induction on k that they coincide. The jets of the coefficients of L and L_W at 0 must therefore coincide too. This gives us the first half of the following result.

Proposition **5.22.**

(i) *If the series* (5.21) *converge in a neighbourhood of* $z = \infty$, *then there are meromorphic functions* $\widetilde{u}_0, \dots, \widetilde{u}_{n-2}$ *defined in the entire complex plane such that* u_i *and* $\widetilde{u}_i$ *have the same Taylor series at* $x = 0$.

(ii) *If the series* $\psi(x, z)$ *converges for* $|z| > R$ *for each* x *in* I *then the* u_i *coincide with the* $\widetilde{u}_i$ *in* I.

To prove the second statement, let $\varphi_i(x, z^n)$ be the solution of $L\varphi = z^n\varphi$ characterized by the initial conditions $D^j\varphi_i(0, z^n) = \delta_{ij}$. Each φ_i is an entire function of z^n for x in I. If z is fixed and $|z| > R$, then $\psi(x, z)$ and

$$\sum_{i=0}^{n-1}\varphi_i(x, z^n)\, D^i\psi(0, z)$$

are both solutions of $L\varphi = z^n\varphi$ with the same initial conditions. They must therefore coincide, and it follows that $\psi(x, \cdot)$ belongs to W for all $x \in I$. As $\psi(x, \cdot)$ also belongs to $e^{xz}(1 + H_-)$, we can conclude that $\psi(x, z) = \psi_W(x, z)$ for all $x \in I$, and so L and L_W coincide in I.

6. Algebraic curves: the construction of Krichever

In Krichever's construction of solutions to the KdV equations the starting point is a collection of data whose most important constituents are a compact Riemann surface X and a holomorphic line bundle $\mathscr{L}$ over it. Mumford [16] pointed out that the construction still applies more or less unchanged if we allow X to be any complete irreducible complex algebraic curve (possibly singular), and that in that case it is natural to allow $\mathscr{L}$ to be, more generally, a rank 1 torsion free coherent sheaf over X. (If X is non-singular, any such sheaf is a line bundle.) One reason for including singular curves is that the n-soliton solutions correspond to rational curves with n double points; and even the solutions coming from torsion free sheaves that are not line bundles seem to have nothing very exotic about them (we shall see examples in § 7). The inclusion of torsion free sheaves will not cause us any extra difficulty, and will be essential for the proof of theorem 6.10 below.

As well as X and $\mathscr{L}$, the construction requires three more pieces of data (x_∞, z, φ). Here x_∞ is a non-singular point of X and z^{-1} is a local parameter on X near x_∞. We shall suppose that z is an isomorphism from some closed neighbourhood X_∞ of x_∞ in X to the disc $D_\infty = \{|z| \geq 1\}$ in the Riemann sphere. That can always be achieved by rescaling z (see remark 6.5 below). Finally, φ is a trivialization of $\mathscr{L}$ over X_∞. We shall use φ to identify sections of $\mathscr{L}$ over subsets of X_∞ with complex-valued functions. We shall also identify the unit circle S^1 with its inverse image in X under z. We denote by X_0 the complement of the interior of X_∞: thus the closed sets X_∞ and X_0 cover X, and their intersection is S^1.

To all this data we associate the following subspace W of $H = L^2(S^1, \mathbf{C})$: W *is the closure of the space of analytic functions on* S^1 *that extend to sections of* $\mathscr{L}$ *over* X_0.

Proposition **6.1.** — *The subspace* W *belongs to the Grassmannian* Gr. *The virtual dimension of* W *is equal to* $\chi(\mathscr{L}) - 1$, *where as usual* $\chi(\mathscr{L})$ *denotes the Euler characteristic* $\dim H^0(X; \mathscr{L}) - \dim H^1(X; \mathscr{L})$.

Proof. — We observe first that the projection $W \to H_-$ factorizes

$$W \xrightarrow{R_{\lambda^{-1}}} H \xrightarrow{\text{pr}} H_- \xrightarrow{R_\lambda} H_-$$

for suitable λ with $0 < \lambda < 1$ (here R_λ is the scaling transformation discussed in § 2). For λ sufficiently close to 1, the map $R_{\lambda^{-1}}: W \to H$ is bounded: for each $f \in W$ is the boundary value of a holomorphic section of $\mathscr{L}$ over $X \setminus X_\infty$, and (by assumption) the trivialization φ extends over some open set containing X_∞. Thus $R_{\lambda^{-1}}$ simply assigns to $f \in W$ the function $z \mapsto f(\lambda z)$, i.e. f evaluated on a circle slightly inside the boundary of X_0. Since $R_\lambda : H_- \to H_-$ is compact, the projection $W \to H_-$ is too. It follows easily that the projection $W \to H_+$ has closed range.

It remains to show that the projection $W \to H_+$ is a Fredholm operator of the index stated. We shall prove a more precise statement: the kernel and cokernel of the orthogonal projection $W \to zH_+$ are $H^0(X, \mathscr{L})$ and $H^1(X, \mathscr{L})$ respectively. Let U_0 and U_∞ be open sets of X containing X_0 and X_∞, and let $U_{0\infty} = U_0 \cap U_\infty$. Because U_0, U_∞, and $U_{0\infty}$ are Stein varieties, we can calculate the cohomology of X with coefficients in any coherent sheaf from the covering $\{U_0, U_\infty\}$; in particular, we have an exact sequence

$$0 \to H^0(X, \mathscr{L}) \to \mathscr{L}(U_0) \oplus \mathscr{L}(U_\infty) \to \mathscr{L}(U_{0\infty}) \to H^1(X, \mathscr{L}) \to 0,$$

where $\mathscr{L}(U)$ denotes the sections of $\mathscr{L}$ over a subset U of X. Taking the direct limit of this as U_0 and U_∞ shrink to X_0 and X_∞ gives the exact sequence

$$0 \to H^0(X, \mathscr{L}) \to \mathscr{L}(X_0) \oplus \mathscr{L}(X_\infty) \to \mathscr{L}(S^1) \to H^1(X, \mathscr{L}) \to 0.$$

Since $\mathscr{L}$ is torsion free, its sections over X_0 or X_∞ are determined by their restrictions to S^1; thus we can identify $\mathscr{L}(X_0)$ and $\mathscr{L}(X_\infty)$ with subspaces of the space $\mathscr{L}(S^1)$ of real analytic functions on S^1. The two middle terms in the above exact sequence then become

$$W^{an} \oplus zH^{an}_- \to H^{an},$$

the map being the inclusion on the first factor and minus the inclusion on the second factor (we write V^{an} for the analytic functions in a subspace V of H). The kernel and cokernel of this map are the same as those of the projection $W^{an} \to zH^{an}_+$, so we have only to see that the kernel and cokernel of this do not change when we pass to the completions $W \to zH_+$. But a function in the kernel of this last projection is the common L^2 boundary value of holomorphic functions defined inside and outside S^1, hence it must be analytic: thus the two kernels coincide. That the cokernels coincide too follows easily from the fact that $W \to H_+$ has closed range.

The same argument shows that the kernel and cokernel of the orthogonal projection $W \to H_+$ can be identified with $H^0(X, \mathscr{L}_\infty)$ and $H^1(X, \mathscr{L}_\infty)$, where $\mathscr{L}_\infty = \mathscr{L} \otimes [-x_\infty]$ is the sheaf whose sections are sections of $\mathscr{L}$ that vanish at x_∞. In particular, W is transverse if and only if we have $H^0(X, \mathscr{L}_\infty) = H^1(X, \mathscr{L}_\infty) = 0$. For readers of [16, 21], we note that it is the sheaf $\mathscr{L}_\infty$, rather than $\mathscr{L}$, that is considered in those papers.

We are mainly interested in spaces of virtual dimension zero; by (6.1), these arise from sheaves with $\chi(\mathscr{L}) = 1$. If $\mathscr{L}$ is a line bundle, the Riemann-Roch theorem shows that its degree is then the arithmetic genus of X.

Combining the construction above with that of § 5, we obtain a solution to the KP equations for each set of data $(X, x_\infty, z, \mathscr{L}, \varphi)$ with $\chi(\mathscr{L}) = 1$. This construction is essentially the same as that of Krichever [10, 11]. To be more precise, Krichever considers the case where X is non-singular, and starts off from a positive divisor $\mathscr{D} = \{P_1, \ldots, P_g\}$, with $P_i \in X$, of degree g equal to the genus of X. He assumes that no P_i is the point x_∞, and that $\mathscr{D}$ is non-special. "Non-special" means that the line bundle $\mathscr{L}$ corresponding to $\mathscr{D}$ has a unique (up to a constant multiple) holomorphic section, which vanishes precisely at the points P_i; this section therefore defines a trivialization of $\mathscr{L}$ over the complement of $\{P_i\}$, in particular over a neighbourhood of x_∞.

If all the points P_i lie outside the disc X_∞, we can use this trivialization; our construction then reduces exactly to Krichever's.

The correspondence that we have described between algebro-geometric data and subspaces of H is obviously not one to one, for the following reason: suppose $\pi : X' \to X$ is a map which is a birational equivalence (that is, intuitively, the curve X is obtained from X' by making it " more singular "). Then we obtain the same space W from a sheaf $\mathscr{L}'$ on X' and from its direct image $\mathscr{L} = \pi_*(\mathscr{L}')$ on X. We shall avoid this ambiguity by agreeing to consider only *maximal* torsion free sheaves on X, that is ones that do not arise as the direct image of a sheaf on a less singular curve. A perhaps more illuminating description of them is as follows. Recall (see [7]) that the rank 1 torsion free sheaves over X (of some fixed Euler characteristic) form a compact moduli space M on which the generalized Jacobian of X (the line bundles of degree zero) acts by tensor product. We claim that the maximal torsion free sheaves form precisely the part of M on which the Jacobian acts freely. Indeed, if $\mathscr{L}$ is any rank 1 torsion free sheaf on X and L is a line bundle of degree zero, then giving an isomorphism $L \otimes \mathscr{L} \cong \mathscr{L}$ is equivalent to giving an isomorphism $L \cong \mathit{Hom}(\mathscr{L}, \mathscr{L})$; but $\mathit{Hom}(\mathscr{L}, \mathscr{L})$ is just the structure sheaf of the " least singular " curve X' such that $\mathscr{L}$ is the direct image of a sheaf on X', hence it is $\mathcal{O}_X$ exactly when $\mathscr{L}$ is maximal. Obviously, any line bundle is a maximal torsion free sheaf; and if all the singularities of X are planar, these are the only ones, for in that case (and only in that case) M is an irreducible variety containing the line bundles as a Zariski open subset (see [34]). However, in general there are many maximal torsion free sheaves that are not line bundles: we shall meet simple examples in § 7.

Proposition **6.2.** — *The construction described above sets up a one to one correspondence between isomorphism classes of data* $(X, \mathscr{L}, x_\infty, z, \varphi)$, *with* $\mathscr{L}$ *maximal, and certain spaces* $W \in \mathrm{Gr}$.

Proof. — Let W be the space arising from data $(X, \mathscr{L}, x_\infty, z, \varphi)$ with $\mathscr{L}$ maximal. We have to show how to reconstruct all of this data (up to isomorphism) from W alone. Let us recall from (2.6) the definition of the dense subspace W^{alg} of W, consisting of all elements of finite order. Clearly W^{alg} can be identified with the space of algebraic sections of $\mathscr{L}$ over $X \setminus \{x_\infty\}$. If A is the coordinate ring of the affine curve $X \setminus \{x_\infty\}$, then W^{alg} is the rank one torsion free A-module corresponding to the sheaf $\mathscr{L}$ restricted to $X \setminus \{x_\infty\}$. On the other hand, let A_W be the ring of analytic functions f on S^1 such that $f.W^{alg} \subset W^{alg}$. Clearly A_W is an algebra containing A (if we identify functions in A with their restrictions to S^1), and W^{alg} is a faithful A_W-module. As W is torsion-free and of rank one as a module over A, it follows that A_W can be identified with an integral subring of the quotient field of A. This means that $\mathrm{Spec}(A_W)$ is a curve of the form $X' \setminus \{x_\infty\}$ (with X' complete) projecting birationally on to $X \setminus \{x_\infty\}$; and so if $\mathscr{L}$ is maximal we must have $A_W = A$. Thus we have reconstructed from W the curve X, the point x_∞, and the restriction of $\mathscr{L}$ to $X \setminus \{x_\infty\}$. Finally, the inclusion

$W^{alg} \subset \mathbf{C}[z] \oplus H_-$ defines a trivialization of $\mathscr{L}$ over $X_\infty \setminus \{x_\infty\}$ (and hence the extension of $\mathscr{L}$ to X); for if $|\zeta| > 1$ then evaluation at ζ defines a map $W^{alg} \to \mathbf{C}$ which induces an isomorphism of the fibre of $\mathscr{L}$ at ζ with $\mathbf{C}$. (That is clear, because the fibre is canonically $W^{alg}/\mathfrak{m}W^{alg}$, where $\mathfrak{m} \subset A_W$ is the ideal of functions that vanish at ζ.)

Remark **6.3.** — The definition of A_W makes sense for any $W \in Gr$. In general, however, it will be trivial, i.e. $A_W = \mathbf{C}$. (This is clearly the case, for example, when W is the subspace of codimension one in H_+ which was described in (2.10).) The spaces $W \in Gr$ which arise from algebro-geometrical data are precisely those such that A_W contains an element of each sufficiently large order, or, equivalently, such that the A_W-module W^{alg} has rank 1. That follows at once from the preceding discussion, in view of the fact that the coordinate rings A of irreducible curves of the form $X \setminus \{x_\infty\}$ (where X is complete and x_∞ is a non-singular point) are characterized as integral domains simply by the existence of a filtration

$$\mathbf{C} = A_0 \subset A_1 \subset A_2 \subset \ldots \subset A$$

such that

(i) $A_i . A_j \subset A_{i+j}$,

(ii) $\dim(A_k/A_{k-1}) \leq 1$ for all k, and

(iii) $\dim(A_k/A_{k-1}) = 1$ for all large k.

Remark **6.4.** — We should point out that for any $W \in Gr$ the construction of § 5 defines a realization of A_W as a commutative ring of differential operators. More precisely, the proof of (5.11) shows that for any $f \in A_W$ there is a unique differential operator $L(f)$ such that

$$L(f)\, \psi_W = f(z)\, \psi_W.$$

If $W \in Gr^{(n)}$, then $z^n \in A_W$, and $L(z^n) = L_W$. In general, the order of the operator $L(f)$ is equal to the order of f.

Remark **6.5.** — As we saw in § 5, a change of local parameter $z \mapsto cz$ (c a non-zero constant) corresponds to acting on the solution to the KP hierarchy by the scaling transformation. Thus the condition that the validity of the parameter z should extend up to $|z| = 1$ is not a serious restriction in our theory.

Remark **6.6.** — The solution to the KP hierarchy does not depend on the choice of trivialization φ: for a different choice of φ would just multiply W by a function of the form $c_0 + c_1 z^{-1} + \ldots$ (with $c_0 \neq 0$), which, as we know, does not change the solution. Even the space W does not change if we replace φ by $c\varphi$ where c is a non-zero constant; that does not contradict (6.2), because the quintuples of data $(X, x_\infty, z, \mathscr{L}, c\varphi)$ for different $c \neq 0$ are obviously isomorphic.

Remark **6.7.** — We get a solution to the n-th KdV hierarchy (i.e. $W \in \mathrm{Gr}^{(n)}$) if and only if $z^n \in A_W$; that is, if the local parameter z is such that z^n extends to a meromorphic function on X with no singularities except for the n-th order pole at x_∞. For fixed n, this of course imposes a restriction on which pairs (X, x_∞) can occur: for example if $n = 2$ then X must be hyperelliptic and x_∞ must be a Weierstrass point.

Remark **6.8.** — An important part of Krichever's theory is the observation that the KdV (or KP) flows correspond to straight line motion on the Jacobian of X. That is easily seen from our point of view as follows. For each $g \in \Gamma_+$, let L_g be the line bundle obtained by taking trivial bundles over X_0 and X_∞ and glueing them by the transition function g on (an open neighbourhood of) S^1. Thus L_g comes equipped with a trivialization φ_g over X_∞. The natural action of Γ_+ on Gr corresponds to the following action on the data $(X, x_\infty, z, \mathscr{L}, \varphi)$: $g \in \Gamma_+$ acts trivially on the first three components, and on $(\mathscr{L}, \varphi)$ by tensoring with (L_g, φ_g). The action of Γ_+ on solutions to the KP hierarchy thus corresponds simply to $\mathscr{L} \mapsto \mathscr{L} \otimes L_g$. The assertion about straight line motion is now clear in view of the following result.

Proposition **6.9.** — *The assignment* $g \mapsto L_g$ *defines a surjective homomorphism from* Γ_+ *to the generalized Jacobian of* X *(which consists of all holomorphic line bundles on* X *of degree zero).*

Proof. — If L is a line bundle on X then $L \mid X_0$ and $L \mid X_\infty$ are trivial, for all bundles on affine curves are analytically trivial, and X_0 and X_∞ are contained in affine open sets of X. So $L = L_g$ for some holomorphic function $S^1 \to \mathbf{C}^\times$ whose winding number is the degree of L. We can change g by any element of Γ_- without affecting L_g; and so if g has winding number zero we can choose it in Γ_+.

Example

Let us return briefly to the subspace $W = W_{\mathbf{p},\lambda} \in \mathrm{Gr}_1^{(2)}$ which was introduced in § 3 and discussed further in § 5. In this case A_W consists of all polynomials f in z such that $f(-p_i) = f(p_i)$ for each i. This is the coordinate ring of the affine curve whose completion $X_\mathbf{p}$ is obtained from the Riemann sphere by identifying the point p_i with $-p_i$ for each i: $X_\mathbf{p}$ is a rational curve with m double points. If we take $\xi = z^2$ and $\eta = z(z^2 - p_1^2) \ldots (z^2 - p_m^2)$ as generators of A_W then the equation of $X_\mathbf{p}$ is

$$\eta^2 = \xi(\xi - p_1^2)^2 \ldots (\xi - p_m^2)^2.$$

We remarked in § 5 that the orbit of $W_{\mathbf{p},\lambda}$ under Γ_+ consists of all $W_{\mathbf{p},\mu}$, where μ runs through $(\mathbf{C}^\times)^m$. This conforms with (6.9), as $(\mathbf{C}^\times)^m$ is the generalized Jacobian of $X_\mathbf{p}$.

Commuting differential operators

Our last goal in this section is to point out that our results lead directly to a proof of the so-called "Painlevé property" of the stationary KdV equations. Since these have the form $[P, L] = 0$, the result can be formulated as a statement about commuting differential operators.

Theorem **6.10.** — *Let* $L = D^n + u_{n-2} D^{n-2} + \ldots + u_0$ *be an ordinary differential operator whose coefficients* u_i *are defined and smooth in a neighbourhood* I *of the origin in* $\mathbf{R}$. *Suppose that there exists a differential operator* P *of order m relatively prime to n that commutes with* L. *Then the functions* u_i *extend to meromorphic functions on the whole complex plane, with poles of order at most* $n - i$, *so that all the finite singular points of* L *are regular.*

Note that the condition about relatively prime orders is obviously essential: if we omitted it there would be trivial counterexamples to the theorem where $L = P$, or more generally L and P are both polynomials in some operator of lower order.

It is easy to see (for example by conjugating L into D^n by a formal integral operator as in § 4) that any operator P that commutes with L is some linear combination

$$P = \sum_0^N c_r L_+^{r/n}$$

of the operators P_r occuring in the definition of the n-th KdV hierarchy. For each fixed sequence of constants $\{c_r\}$, the stationary KdV equation $[P, L] = 0$ is a system of ordinary differential equations for the coefficients $\{u_0, \ldots, u_{n-2}\}$ of L. Let us call such an equation (or the corresponding P) *admissible* if there is some index r relatively prime to n with $c_r \neq 0$. For example, if n is prime, then every non-trivial stationary KdV equation is admissible. If P is admissible, then the algebra generated by L and P contains operators of order relatively prime to n. Thus (6.10) can be formulated as follows: *every solution* $\{u_i\}$ *of an admissible stationary KdV equation is of the kind stated in the conclusion of* (6.10).

Theorem 6.10 will follow from (5.18) if we show that every operator L satisfying the hypotheses is of the form L_W for some $W \in Gr^{(n)}$ arising from an algebraic curve. This is well known, and is proved in [16, 21]; however, for completeness we give a self-contained proof, following the approach of Burchnall and Chaundy [4].

Proposition **6.11.** — *If* L *and* P *are commuting differential operators as in* (6.10), *then:*

(i) *There is an irreducible polynomial* $F \in \mathbf{C}[\xi, \eta]$ *of the form*

$$F = \xi^m + \ldots \pm \eta^n$$

such that $F(L, P) = 0$.

(ii) *For all but a finite number of points* (λ, μ) *of the affine curve* X_F *whose equation is* $F(\lambda, \mu) = 0$ *there is a unique common eigenfunction* $\varphi_{\lambda,\mu}$ *of* L *and* P *such that* $\varphi_{\lambda,\mu}(0) = 1$:

$$L\varphi_{\lambda,\mu} = \lambda\varphi_{\lambda,\mu}, \qquad P\varphi_{\lambda,\mu} = \mu\varphi_{\lambda,\mu}.$$

For any $x \in I$, $\varphi_{\lambda,\mu}(x)$ *is a meromorphic function on the curve* X_F.

(iii) *For* $x \in \mathrm{I}$ *the formal Baker functions* $\psi_{\mathrm{L}}(x, z)$ *and* $\psi_{\mathrm{P}}(x, z)$ *of* L *and* P *both converge for large* z, *and then*

$$\psi_{\mathrm{L}}(x, \lambda^{1/n}) = \psi_{\mathrm{P}}(x, \mu^{1/m}) = \varphi_{\lambda, \mu}(x).$$

(Notice that $\lambda^{1/n}$ *and* $\mu^{1/m}$ *are local parameters at the point at infinity of* X_{F}.*)*

We begin by proving assertion (i). For any $\lambda \in \mathbf{C}$ let V_λ be the n-dimensional vector space of solutions of $\mathrm{L}\varphi = \lambda\varphi$ on I. A basis for V_λ is given by the functions $\varphi_i(x, \lambda)$ for $0 \leq i < n$ such that $\varphi_i^{(j)}(0, \lambda) = \delta_{ij}$. Notice that for any $\varphi \in \mathrm{V}_\lambda$ and any k we have

$$\varphi^{(k)}(0) = \sum_{i=0}^{n-1} p_{ki}(\lambda)\, \varphi^{(i)}(0)$$

where the $p_{ki}(\lambda)$ are polynomials independent of φ.

The operator P maps V_λ into itself. In terms of the basis $\{\varphi_i\}$ the action of P on V_λ is given by an $n \times n$ matrix P_λ of polynomials in λ. Let $\mathrm{F}(\lambda, \mu)$ be the characteristic polynomial $\det(\mu - \mathrm{P}_\lambda)$. It is not hard to see that $\mathrm{F}(\lambda, \mu)$ is a polynomial of degree m in λ: in fact one can show that (up to sign) it is the same as the polynomial obtained by reversing the roles of P and L in the construction. Thus F has the form stated in (i). Consider the differential operator $\mathrm{F}(\mathrm{L}, \mathrm{P})$. There is at least one solution of $\mathrm{F}(\mathrm{L}, \mathrm{P})\,\varphi = 0$ in each V_λ. As a differential operator can have only finitely many linearly independent solutions, this implies that $\mathrm{F}(\mathrm{L}, \mathrm{P}) = 0$. But the same argument shows that if G is any factor of F then $\mathrm{G}(\mathrm{L}, \mathrm{P}) = 0$; so F must be a power of an irreducible polynomial. As $\mathrm{F}(\lambda, \mu)$ contains the monomials λ^m and μ^n, the power must divide both n and m. But these are relatively prime, so F must be irreducible.

We next prove assertion (ii). Because the polynomial F is irreducible there are, for all but finitely many values of λ, n distinct solutions μ of $\mathrm{F}(\lambda, \mu) = 0$. For each of these values of μ there is (up to a scalar multiple) a unique eigenvector $\varphi_{\lambda, \mu}$ of P_λ in V_λ with eigenvalue μ. We can choose it so that its coordinates with respect to the basis $\{\varphi_i\}$ of V_λ (i.e. its derivatives at 0) are polynomials in λ and μ: for example we can take the coordinates to be the cofactors of any row of the matrix $\mu - \mathrm{P}_\lambda$. The value of $\varphi_{\lambda, \mu}$ at 0 cannot vanish identically, for the eigenvectors of P_λ must span V_λ for almost all λ. This permits us to normalize $\varphi_{\lambda, \mu}$ so that $\varphi_{\lambda, \mu}(0) = 1$, except at a finite number of points (λ, μ). The derivatives $\varphi_{\lambda, \mu}^{(i)}(0)$ will then be rational functions of λ and μ.

To see that $\varphi_{\lambda, \mu}(x)$ is meromorphic on X_{F} we observe that

$$\varphi_{\lambda, \mu}(x) = \sum_{i=0}^{n-1} \varphi_{\lambda, \mu}^{(i)}(0)\, \varphi_i(x, \lambda).$$

(Note that $\varphi_i(x, \lambda)$ is an entire function of λ.)

To prove (iii) we first observe that not only do we have $\mathrm{L}\psi_{\mathrm{L}} = z^n \psi_{\mathrm{L}}$ by definition, but also $\mathrm{P}\psi_{\mathrm{L}} = \mu(z)\, \psi_{\mathrm{L}}$, where $\mu(z)$ is a formal power series belonging to the field $\mathbf{C}((z^{-1}))$ of formal series of the form $\sum_{i=-\infty}^{N} \alpha_i z^i$. To see this, choose K as in § 4 so that $K^{-1} L K = D^n$. Then $K^{-1} P K$ commutes with D^n, and so must be a formal pseudodifferential operator $\mu(D)$ with constant coefficients. Thus $PK = K\mu(D)$. Applying both sides to e^{xz} gives $P\psi_L = \mu(z)\,\psi_L$, as $Ke^{xz} = \psi_L$.

Now we adopt the following point of view. The operators L and P can be thought of as acting on the vector space of jets of functions (of x) at the origin: in other words we replace functions φ by sequences $\{\varphi^{(i)}(0)\}_{i\geq 0}$. Consider the vector space J of formal jets whose components $\varphi^{(i)}(0)$ belong to the field $\mathbf{C}((z^{-1}))$ of formal series. The operator $L - z^n$ acts on J, and has an n-dimensional kernel J_L spanned by the jets of the functions $\varphi_i(x, z^n)$ already mentioned. (Recall that $\varphi_i^{(j)}(0, z^n)$ is a polynomial in z^n.) Now formal series of the form $e^{xz} \Sigma a_k(x)\, z^{-k}$ define jets in J, and the jet of ψ_L belongs to J_L. Furthermore P preserves J_L, and $P\psi_L = \mu(z)\,\psi_L$. On the other hand we already know that the unique eigenvectors of P in J_L, when normalized at 0, are the jets of $\varphi_{\lambda,\mu}$, where $\lambda = z^n$, and μ runs through the n roots of $F(\lambda, \mu) = 0$, which are distinct for large λ. This proves that $\psi_L^{(i)}(0, z) = \varphi^{(i)}_{z^n, \mu(z)}(0)$ for some point $(z^n, \mu(z)) \in X_F$, and hence that the series $\psi_L^{(i)}(0, z)$ and $\mu(z)$ converge for large z.

In the preceding discussion the role of the origin could have been played by any point $x_0 \in I$. Thus we can conclude that if a formal Baker function $\psi_{L,x_0}(x, z)$ is calculated at x_0 then

$$\psi^{(i)}_{L,x_0}(x_0, z) = \varphi^{(i)}_{\lambda,\mu}(x_0)\, e^{x_0 z}\, \varphi_{\lambda,\mu}(x_0)^{-1}.$$

(The factor $e^{x_0 z}\,\varphi_{\lambda,\mu}(x_0)^{-1}$ on the right occurs because ψ_{L,x_0} is normalized by $\psi_{L,x_0}(x_0, z) = e^{x_0 z}$.) The space $W_{x_0} \in \mathrm{Gr}^{(n)}$ defined by ψ_{L,x_0} is therefore related to the space W defined by ψ_L by

$$W_{x_0} = e^{x_0 z}\, \varphi_{z^n, \mu(z)}(x_0)^{-1}\, W.$$

But $e^{-x_0 z}\,\varphi_{z^n,\mu(z)}(x_0)$ does not vanish for large z, and so (after scaling, if necessary) it defines an element γ of the group Γ_-. Thus W_{x_0} and W define the same meromorphic differential operator. The jets of its coefficients coincide with those of L at x_0 and 0 respectively. This proves (iii).

Remark **6.12.** — Notice that we have proved that L arises by Krichever's construction from the curve X_F and the torsion-free sheaf $\mathscr{L}$ whose space of sections over $X_F \setminus \{\infty\}$ is the space W^{alg} generated by the $\varphi^{(i)}_{\lambda,\mu}(0)$. In particular, this proves (6.10). It is not hard to show that the fibre of $\mathscr{L}$ at any point (λ, μ) of X_F is canonically isomorphic to the joint (λ, μ)-eigenspace of L and P.

Remark **6.13.** — We believe that theorem 6.10 is " well known " (except possibly for the assertion about the orders of the poles), but our proof seems to be the first complete one available. Krichever [10] noted that " most " of the solutions (that is, the ones coming from non-singular curves X) of the stationary KdV equations are globally meromorphic; our proof is essentially the same as his except that he used the theta function of X where we use the more general τ-function (see § 9 below). It might be interesting to give a direct algebro-geometric proof of the theorem, presumably by introducing suitable " theta functions " for singular curves. However, we note that one would have to define a theta function, not merely for each singular curve, but for each orbit of the Jacobian of such a curve acting on the space of maximal torsion free sheaves.

7. Rational Curves

We recall from § 2 that Gr_1 is the subspace of Gr consisting of spaces W such that

$$pH_+ \subset W \subset q^{-1} H_+$$

for some polynomials p, q, and that p and q can be chosen so that all their roots lie in the region $|z| < 1$.

Proposition **7.1.** — *The construction described in the preceding section gives a one to one correspondence between spaces* $W \in \mathrm{Gr}_1$ *and isomorphism classes of data* $(X, \mathscr{L}, x_\infty, z, \varphi)$ *as in* (6.2) *such that*

(i) X *is a rational curve*

(ii) z *is a rational parameter on* X

(iii) φ *extends to an algebraic trivialization of* $\mathscr{L}$ *over some Zariski open set containing the disc* X_∞.

Before giving the proof, we clarify the term " rational parameter ". The curve X being rational means that there is an algebraic map f from the Riemann sphere to X which is an isomorphism outside the inverse image of the singular set of X. We can choose f so that $f(\infty) = x_\infty$. By a rational parameter z on X we mean the inverse of such a map f in some neighbourhood of x_∞: note that its domain in fact extends to the whole non-singular part of X. Note also that the rational parameter is uniquely determined up to a linear change $z \mapsto az + b$: for any two of them differ by a birational, hence genuine, automorphism of the Riemann sphere preserving ∞, which must be linear.

Proof of (7.1).

(i) Let $W \in \mathrm{Gr}_1$, and let p and q be polynomials as above. Let W^{alg} and A_W be as in the proof of (6.2). Clearly we have

$$pC[z] \subset W^{\mathrm{alg}} \subset q^{-1} C[z], \tag{7.2}$$

from which it follows easily that $pq\mathbf{C}[z] \subset A_W \subset (pq)^{-1} \mathbf{C}[z]$. Since A_W is a ring, we have in fact

$$pq\mathbf{C}[z] \subset A_W \subset \mathbf{C}[z].$$

Thus the inclusion of A_W in $\mathbf{C}[z]$ induces an isomorphism of quotient fields; that shows that the curve $X \setminus \{x_\infty\} = \operatorname{Spec} A_W$ is rational, and that z is a rational parameter on X. From (7.2) it is clear that the A_W-module W^{alg}, and hence also the corresponding sheaf $\mathscr{L}$ on X, has rank 1. It remains to prove (iii). Let $z_0 \in \mathbf{C}$; then evaluation at z_0 gives a map $e(z_0) : W^{alg} \to \mathbf{C}$ which is defined provided that z_0 is not a root of q, and non-zero provided that z_0 is not a root of p. Let P_0 be the point of X corresponding to z_0, and let $\mathfrak{m} \subset A_W$ be its maximal ideal. Then $e(z_0)$ defines a map from the fibre $W^{alg}/\mathfrak{m}W^{alg}$ of $\mathscr{L}$ over P_0 to $\mathbf{C}$, which is an isomorphism provided that z_0 satisfies the two conditions above and that P_0 lies in the open set of X over which $\mathscr{L}$ is a line bundle. That completes the proof that W gives rise to algebro-geometric data of the kind stated in the proposition.

(ii) Conversely, suppose we are given data $(X, \mathscr{L}, x_\infty, z, \varphi)$ of the kind listed in the proposition: we have to see that the corresponding W belongs to Gr_1. Let $B \subset X$ be the finite set of points over which φ is not defined. If necessary we enlarge B to include all the singular points of X. Let $\{z_1, \ldots, z_r\} \subset \mathbf{C}$ be the inverse image of B under the map $f: S^2 \to X$ whose inverse is the parameter z. Then we can identify the sections of $\mathscr{L}$ over $X \setminus B$ with the sections of a trivialized line bundle over $S^2 \setminus \{z_1, \ldots, z_r\}$. Thus W^{alg}, which is the space of sections of $\mathscr{L}$ over $X \setminus \{x_\infty\}$, is identified with a subspace of the space $F(z_1, \ldots, z_r; -\nu_1, \ldots, -\nu_r)$ of rational functions of z that are holomorphic except for poles of prescribed orders ν_i at the points z_i. More precisely, W^{alg} is the subspace of $F(z_i, -\nu_i)$ consisting of all functions whose Laurent series at the points z_i satisfy some finite set of linear conditions. These conditions are certainly satisfied by all polynomials that vanish to suitably high orders μ_i at the points z_i. It follows that if we set $p = \Pi(z - z_i)^{\mu_i}$, $q = \Pi(z - z_i)^{\nu_i}$, then we have

$$p\mathbf{C}[z] \subset W^{alg} \subset q^{-1}\,\mathbf{C}[z].$$

Passing to the L^2 closures, we find $pH_+ \subset W \subset q^{-1}H_+$, as required.

Next recall that Gr_0 is the subspace of Gr_1 consisting of spaces W for which the polynomials p and q can be taken to be powers of z. If we follow through the above proof in that case, we obtain the following.

Proposition **7.3.** — *The construction described in* § 6 *gives a one to one correspondence between spaces* $W \in Gr_0$ *and isomorphism classes of data* $(X, x_\infty, z, \mathscr{L}, \varphi)$ *as in* (6.2) *such that*

(i) X *is a rational curve with just one irreducible (i.e. cusp-like) singularity*
(ii) z *is a rational parameter on* X *such that the singular point* x_0 *corresponds to* $z = 0$
(iii) φ *extends to an algebraic trivialization of* $\mathscr{L}$ *over the whole non-singular part* $X \setminus \{x_0\}$ *of* X.

The term " irreducible " in (i) means that when we resolve the singularity we still get only one point, so that z is in fact a bijection between X and the Riemann sphere. Note that z and φ are now both uniquely determined up to multiplication by non-zero constants. The fact that φ is unique means that the correspondence between spaces in Gr_0 and solutions to the KP hierarchy is one to one. Indeed it is easy to see directly that if $W \in Gr_0$ and γ is a function of the form $1 + c_1 z^{-1} + \ldots$, then γW cannot belong to Gr_0 unless $\gamma = 1$.

The subspaces $W \in Gr_0$ provide many simple examples of maximal torsion free sheaves that are not line bundles. Indeed let $W = H_S$, where $S \subset \mathbf{Z}$ is a set of virtual cardinal zero. Then $W \in Gr_0$, and we claim that the corresponding maximal torsion free sheaf is seldom a line bundle. Here W^{alg} is the vector space spanned by $\{z^s\}_{s \in S}$. Let R be the semi-group of strictly positive integers r such that $S + r \subset S$. Then A_W is the algebra spanned by 1 and $\{z^r\}_{r \in R}$, and the maximal ideal $\mathfrak{m}$ of A_W corresponding to the singular point $z = 0$ is spanned by $\{z^r\}_{r \in R}$. The dimension of the fibre $W^{alg}/\mathfrak{m}W^{alg}$ of the sheaf $\mathscr{L}$ over the singular point is thus the number of elements of $S \setminus S'$, where we have set

$$S' = \bigcup_{r \in R} (S + r).$$

Unless this number is 1, the maximal torsion free sheaf $\mathscr{L}$ is not a line bundle. The simplest example is when $S = \{-1, 0, 2, 3, \ldots\}$; then $R = \{3, 4, 5, \ldots\}$ and $S' = \{2, 3, \ldots\}$. In this case the dimension of the singular fibre of $\mathscr{L}$ is 2. Note that since the algebra $A_W = \mathbf{C}[z^3, z^4, z^5]$ needs more than two generators, the singularity here is not planar: this conforms with our observation in § 6 that in the planar case every maximal torsion free sheaf is a line bundle.

The case $n = 2$

In general, the isomorphism classes of data listed in (7.3) are hard to classify. However, if we confine ourselves to the case of $Gr_0^{(2)}$, then many simplifications take place: perhaps the most important is that the orbits of the group Γ_+ in $Gr_0^{(2)}$ coincide with the cells in the cell decomposition described in § 2. Here we give a brief description of the situation, leaving most of the (easy) proofs to the reader. For simplicity, what follows will refer only to the component of $Gr_0^{(2)}$ consisting of spaces of virtual dimension zero.

We recall from § 2 that $Gr_0^{(2)}$ has a cell decomposition with cells indexed by the sets $S \in \mathscr{S}$ such that $S + 2 \subset S$. It is easy to see that the only such S are the sets S_k given by

$$S_k = \{-k, -k+2, -k+4, \ldots, k, k+1, k+2, \ldots\}.$$

We denote by C_k the corresponding cell in $Gr_0^{(2)}$; it has complex dimension k, and consists of all W of virtual dimension zero such that $z^k H_+ \subset W \subset z^{-k} H_+$ and k is the smallest number with this property. It is not hard to see directly that these W form a k-dimensional cell: such a space W contains elements w of the form

$$w = z^{-k} + \alpha_1 z^{-k+1} + \ldots + \alpha_{2k-1} z^{k-1},$$

and $\{w, z^2 w, \ldots, z^{2k-2} w\}$ is then a basis for $W/z^k H_+$. Thus w determines W uniquely. The converse is not true; however, the coefficients α_i can be normalized in various ways, of which the most convenient for us is the following.

Lemma **7.4.** — *Each* $W \in C_k$ *contains a unique element* w *of the form*

$$w = z^{-k} \exp(a_1 z + a_2 z^3 + \ldots + a_k z^{2k-1}).$$

The correspondence $W \leftrightarrow (a_1, \ldots, a_k)$ gives us an explicit isomorphism of the cell C_k with $\mathbf{C}^k$; the centre of the cell (corresponding to the origin in $\mathbf{C}^k$) is the space H_{S_k}, which we shall denote simply by H_k. It is clear from (7.4) that the subgroup $\{\exp(tz^{2r-1})\}$ of Γ_+ acts on C_k by translating the r-th coordinate a_r. In particular, we see that C_k is precisely the orbit of H_k under Γ_+.

It is interesting to see how this description of the orbits of Γ_+ fits in with the algebro-geometric one implicit in (7.3). The main points are as follows. First, if $W = H_k$ then $A_W = \mathbf{C}[z^2, z^{2k+1}] = A_k$, say. Let $X_k = (\text{Spec } A_k) \cup \{x_\infty\}$ be the corresponding complete curve, and let J_k be the Jacobian of X_k (parametrizing line bundles of degree zero). If we use the point x_∞ to identify the spaces of line bundles of different degrees, then the torsion free sheaf over X_k corresponding to the space H_k is the neutral element in J_k; indeed, it is clear that $H_k^{\text{alg}} = z^{-k} A_k$. Hence the orbit of H_k under Γ_+, that is, the cell C_k, can be identified with the Jacobian J_k. The fact that the cells C_k exhaust $\mathrm{Gr}_0^{(2)}$ implies that the curves X_k are the only ones that arise from a space $W \in \mathrm{Gr}_0^{(2)}$, and also that every maximal torsion free sheaf over one of the curves X_k is a line bundle. Both of these facts can be seen directly: it is easy to show that the A_k are the only subalgebras of $\mathbf{C}[z]$ containing z^2 and also an odd power of z; and, as we have observed before, the assertion about the sheaves is true for any curve with planar singularities (a simple proof that covers our present case (X degenerate hyperelliptic) can be found in [8]; in fact the assertion for singularities of the type $y^n = x^m$ is implicitly contained in [4 (c)]). To see directly that J_k is a k-dimensional cell, we can use the exponential sheaf theory exact sequence: since $H^1(X_k, \mathbf{Z}) = 0$, this gives an isomorphism $H^1(X_k, \mathcal{O}) \cong J_k$. The dimension k of the vector space $H^1(X_k, \mathcal{O})$ can be calculated as the number of " gaps " in the ring A_k, that is, the number of positive integers r such that A_k does not contain a polynomial of order r. The algebras A_k are invariant under $z \mapsto cz$, which implies that the pairs (X_k, cz) for different $c \neq 0$ are isomorphic, so that the scaling transformations can be viewed as flows on the Jacobians J_k. Indeed, from (7.4) we see that the scaling flow on the cell C_k is given by

$$R_\lambda(a_1, \ldots, a_k) = (\lambda^{-1} a_1, \ldots, \lambda^{-2k+1} a_k).$$

Finally, it is interesting to consider the closure $\overline{C}_k$ of the cell C_k: this the union of all the cells C_r with $r \leq k$. Alternatively, $\overline{C}_k$ consists of all $W \in \mathrm{Gr}_0^{(2)}$ such that $A_k W \subset W$. Hence each point of $\overline{C}_k$ determines a torsion free sheaf (in general not maximal) over X_k; in fact we get a bijective map $\overline{C}_k \to M_k$, where M_k is the moduli space of rank one torsion free sheaves of some fixed Euler characteristic over X_k (see [7]). The closed cell $\overline{C}_k$ is an algebraic variety, for it is an algebraic subset (given by the condition $z^2 W \subset W$) of the Grassmannian of k-dimensional subspaces of $z^{-k} H_+/z^k H_+$, and it is fairly clear that the above construction gives us an algebraic family of sheaves over X_k: that implies that the bijection $\overline{C}_k \to M_k$ is an algebraic map. Unfortunately, we cannot assert that it is an isomorphism of algebraic varieties: for example, $\overline{C}_1$ is a one-dimensional projective space (non-singular), whereas M_1 is isomorphic to the curve X_1, which has a cusp. (We do not know a precise reference for this fact, but P. Deligne and T. Ekedahl have kindly pointed out to us that it follows easily from (2.6.1) in [24].) In general, we expect that $\overline{C}_k$ is the normalization of M_k. The inclusion $A_k \subset A_{k-1}$ induces a map $X_{k-1} \to X_k$, and hence (taking the direct image of sheaves) a map $M_{k-1} \to M_k$. This map corresponds to the inclusion $\overline{C}_{k-1} \subset \overline{C}_k$, and identifies M_{k-1} with the boundary of M_k, that is, with the space of torsion free sheaves over X_k that are not line bundles.

The solutions to the KdV equations corresponding to the points of $\mathrm{Gr}_0^{(2)}$ have been much studied (see [1, 2]): the cell C_k corresponds to the solutions to the KdV hierarchy flowing out of the initial value

$$u(x, 0, 0, \dots) = -k(k+1)/x^2.$$

(This is the initial value defined by the space H_k, as will become clear in § 8, when we describe the τ-functions of the spaces H_S.) It is known that these exhaust the rational solutions to the KdV hierarchy that vanish at $x = \infty$.

8. The τ-function and Schur functions

We have already given explicit formulae (3.4) and (3.5) for the τ-function as an infinite determinant. It is useful for some purposes to make the formula even more explicit by expanding the determinants in a certain way: the result is that the τ-function can be written as an infinite linear combination of Schur functions.

We begin by reviewing the basic definitions concerning partitions and Schur functions (for more details see, for example, [13]). By a *partition* we mean an infinite sequence $\nu = (\nu_0, \nu_1, \nu_2, \dots)$ of non-negative integers such that $\nu_0 \geq \nu_1 \geq \nu_2 \geq \dots$ and all except a finite number of the ν_i are zero. The number $|\nu| = \Sigma \nu_i$ is called the *weight* of ν. To each partition ν there is associated a *Schur function* F_ν. This is a polynomial with integer coefficients in a sequence of indeterminates $(h_1, h_2, h_3, \dots)$; it is homogeneous of weight $|\nu|$ when h_i is given weight i. One way to define it is as the $r \times r$ determinant

$$F_\nu(\mathbf{h}) = \det(h_{\nu_i - i + j}), \quad (0 \leq i, j \leq r-1)$$

where r is any number sufficiently large so that $\nu_i = 0$ for $i \geq r$. Here it is understood that $h_0 = 1$ and $h_i = 0$ for $i < 0$; it is clear that the value of the determinant does not depend on the choice of r. One reason for the importance of Schur functions is that they are characters of the general linear groups $GL_N(\mathbf{C})$: to each partition ν there corresponds an irreducible representation of $GL_N(\mathbf{C})$ (for any large N), and its character χ_ν is given by $\chi_\nu(A) = F_\nu(\mathbf{h})$, where

$$1 + \sum_1^\infty h_i z^i = \{\det(1 - Az)\}^{-1},$$

that is, the h_i are the "complete homogeneous symmetric functions" of the eigenvalues of the matrix A. In our context, however, the Schur functions arise in a purely formal manner, and the representations of $\mathrm{GL}_N(\mathbf{C})$ do not seem to be relevant.

Let $\mathscr{S}_0$ denote the set of all subsets $S \subset \mathbf{Z}$ of virtual cardinal zero (see § 2); that is, $\mathscr{S}_0$ consists of all strictly increasing sequences $S = \{s_0, s_1, s_2, \dots\}$ of integers such that $s_i = i$ for all except a finite number of indices i.

Lemma **8.1.** — *There is a one to one correspondence between elements of* $\mathscr{S}_0$ *and partitions, given by* $S \leftrightarrow \nu$ *where* $\nu_i = i - s_i$.

The proof is trivial. Notice that the weight $|\nu|$ of a partition is equal to the length $\ell(S)$ of the corresponding S; that is, it is the codimension of the stratum Σ_S of Gr. In what follows we shall write F_S for the Schur function of the partition corresponding to an element $S \in \mathscr{S}_0$.

Recall from § 2 that if $S \in \mathscr{S}_0$, then $H_S \in \mathrm{Gr}$ is the closed subspace of H spanned by $\{z^s\}_{s \in S}$.

Proposition **8.2.** — *Let* $W = H_S$, *where* $S \in \mathscr{S}_0$. *Then the* τ*-function of* W *is given by*

$$\tau_W(g) = F_S(\mathbf{h})$$

where we have set

$$g^{-1} = 1 + \sum_1^\infty h_i z^i.$$

Proof. — We use the formula (3.4). As an admissible basis for H_S, we choose $w_i = z^{s_i}$ where $S = \{s_0, s_1, s_2, \dots\}$. Also, the map $(a, b) : H \to H_+$ is just $f \mapsto (fg^{-1})_+$, where the subscript $+$ denotes orthogonal projection onto H_+. Thus if g^{-1} is expanded as in the statement of the proposition, it follows at once that the matrix of the map $aw_+ + bw_- : H_+ \to H_+$ is

$$(h_{j-s_i}) = (h_{(i-s_i)-i+j}), \quad (i, j \in \mathbf{N}).$$

Since $s_i = i$ for large i, this matrix is strictly (that is, with 1's on the diagonal) upper triangular apart from a finite block in the top left corner. The matrix of the map $a : H_+ \to H_+$ is strictly upper triangular, so it follows easily that the τ-function

$$\det(w_+ + a^{-1} bw_-) = \det a^{-1}(aw_+ + bw_-)$$

is equal to the determinant of this finite block. That proves the proposition.

Now let $W \in \mathrm{Gr}$ be any space of virtual dimension zero. Fix an admissible basis $w = (w_0, w_1, \dots)$ for W. As in § 3, we think of w as a $\mathbf{Z} \times \mathbf{N}$ matrix, using the natural basis $\{z^k\}$ for H. For each $S \in \mathscr{S}_0$, let w^S be the determinant of the $\mathbf{N} \times \mathbf{N}$ matrix formed by extracting from w the rows indexed by the numbers $s \in S$; that is, if $w_j = \Sigma w_{ij} z^i$, we set

$$w^S = \det(w_{ij})_{i \in S, j \in \mathbf{N}}.$$

We call the numbers $\{w^S\}$ the *Plücker coordinates* of W: they are homogeneous coordinates (a different choice of admissible basis for W multiplies them all by the same non-zero constant). As in the finite dimensional case, the Plücker coordinates can be regarded as giving a projective embedding of the Grassmannian (see the appendix § 10 below). Notice that w^S is non-zero precisely when W is transverse to $H_S^\perp$: indeed, w^S is just the determinant of the orthogonal projection $W \to H_S$ with respect to the bases $\{w_j\}$ for W and $\{z^s : s \in S\}$ for H_S. In particular, by (2.5), there is a unique S of minimal length such that $w^S \neq 0$. If we choose w so that w_+ has the form $1 +$ (finite rank), then the w^S reduce to finite determinants. For example, if W is transverse, we can choose w so that $w_+ = 1$, and then if we set $S \setminus \mathbf{N} = A$ and $\mathbf{N} \setminus S = B$ we have

$$w^S = \det(w_{ij})_{i \in A, j \in B}.$$

Proposition **8.3.** — *The τ-function of* W *is given by*

$$\tau_W(g) = \sum_S w^S F_S(\mathbf{h}),$$

where $\{w^S\}$ are the Plücker coordinates of W, *the sum is taken over all* $S \in \mathscr{S}_0$, *and the variables h_i are related to g as in* (8.2).

Proof. — We first observe that if v and w are $m \times n$ and $n \times m$ matrices respectively, with $n \geq m$, then we have

$$\det vw = \Sigma v_S w^S,$$

where the sum is taken over all subsets $S \subset \{1, 2, \ldots, n\}$ with m elements, v_S is the determinant formed from the columns of v indexed by the elements of S, and w^S is the determinant formed from the corresponding rows of w. (This identity simply expresses the functoriality of the m-th exterior power.) It is not hard to see that the identity extends to a product of infinite matrices, indexed by $\mathbf{N} \times \mathbf{Z}$ and $\mathbf{Z} \times \mathbf{N}$, of the form

$$(v_+, v_-) \begin{pmatrix} w_+ \\ w_- \end{pmatrix}$$

where $v_+ - 1$, $w_+ - 1$, v_- and w_- are all of trace class and S runs through the indexing sets $S \subset \mathbf{Z}$ of virtual cardinal zero.

We apply this to the determinant (3.4) giving the τ-function, with

$$(v_+, v_-) = (1, a^{-1} b).$$

Then w^S is the Plücker coordinate defined above and v_S is the τ-function of H_S, which we calculated in (8.2). That completes the proof.

As we saw in § 5, for the application to differential equations we have to write the elements of Γ_+ in the form

$$g(z) = \exp(\sum_1^\infty t_i z^i)$$

(we write t_1 where we wrote x in § 5). We shall write $\tau_W(\mathbf{t})$ for the τ-function expressed in terms of these " coordinates " on Γ_+: to calculate $\tau_W(\mathbf{t})$ from (8.2) or (8.3), we have only to substitute the variables t_i for the h_i, using the relation

$$\textbf{(8.4)} \qquad \exp(-\sum_1^\infty t_i z^i) = 1 + \sum_1^\infty h_i z^i.$$

Each t_k is a polynomial in the h_i, homogeneous of weight k if we give h_i weight i. If we regard the h_i as symmetric functions of the eigenvalues $\{\lambda_j\}$ of a matrix, as above, then the t_k are given by

$$-kt_k = \sum_j \lambda_j^k$$

(this differs by a sign from the convention adopted in [5]).

Example. — Let $S = \{-1, 0, 2, 3, \ldots\}$. The corresponding partition is $\nu = (1, 1, 0, \ldots)$, so the Schur function is

$$F_S(\mathbf{h}) = \det \begin{pmatrix} h_1 & h_2 \\ 1 & h_1 \end{pmatrix} = h_1^2 - h_2.$$

From (8.4), we have $h_1 = -t_1, h_2 = \frac{1}{2}t_1^2 - t_2$, so by (8.2) the τ-function of the space $W = H_S$ is

$$\tau_W(\mathbf{t}) = \frac{1}{2}t_1^2 + t_2.$$

We end this section with some examples of the use of (8.3). First, note that it is clear that W has only finitely many non-zero Plücker coordinates if and only if it belongs to Gr_0; hence we can read off from (8.3) the following.

Proposition **8.5.** — *The function* $\tau_W(\mathbf{t})$ *is a polynomial in (a finite number of) the variables* $(t_1, t_2, \ldots)$ *if and only if* W *belongs to* Gr_0.

As a more substantial application of (8.3), we shall prove the assertion (5.17) about the orders of the poles of the functions $a_i(x, \mathbf{t}^0)$. We shall continue to write t_1 instead of x. The crucial ingredient in the proof is the fact that the restriction of the τ-function to the one parameter subgroup $\exp(t_1 z)$ of Γ_+ cannot be identically zero. More precisely, we have the following.

Proposition **8.6.** — *For any* $W \in \mathrm{Gr}$, *we have*

$$\tau_W(t_1, 0, 0, \ldots) = ct_1^\ell + \textit{(higher terms)},$$

where $c \neq 0$ *and* ℓ *is the codimension of the stratum of* Gr *containing* W (*).

(*) Added in proof. J. Fay has independently proved an equivalent result in the case when W arises from a Riemann surface. (See his paper " On the even-order vanishing of Jacobian theta functions ", *Duke Math. J.*, **51** (1984), 109-132, thm 1.2.)

In particular, the proposition shows that the τ-function cannot vanish identically, a fact that we used implicitly throughout § 5.

Proof of **(8.6).** — We first consider the behaviour of a Schur function F_S when we set $t_2 = t_3 = \ldots = 0$. Since F_S is a homogeneous polynomial of weight $\ell(S)$ in the t_i, it is clear that we have

$$F_S(t_1, 0, 0, \ldots) = d_S\, t_1^{\ell(S)}$$

where d_S is some rational number. We claim that this number is non-zero. Indeed, d_S is equal to $(-1)^{\ell(S)}$ times the reciprocal of a certain positive integer, the "product of the hook lengths" of the partition associated to S (see [13], p. 37, ex. 3). Explicitly, we have

$$(-1)^{\ell(S)}\, d_S = \prod_{i<j\leq n} (s_j - s_i) \Big/ \prod_{i \leq n} (n - s_i)!$$

where n is any number large enough so that $s_{n+1} = n + 1$, and as usual $S = \{s_0, s_1, \ldots\}$ (see [13], p. 9, formula (4)). We have already observed that for any $W \in \mathrm{Gr}$, there is a unique S of minimal length ℓ, say, such that the Plücker coordinate w^S is non-zero; this S is the index of the stratum containing W, and ℓ is the codimension of the stratum. That means that in the expansion (8.3) of τ_W, all the terms have weight at least ℓ; and the terms of minimal weight ℓ form a non-zero multiple of a single Schur function F_S. Thus the proposition follows at once from (8.3) and the fact that $d_S \neq 0$.

Proof of **(5.17).** — Replacing W if necessary by gW for suitable $g \in \Gamma_+$, we see that it is enough to consider the case where the pole is at the origin $\mathbf{t} = 0$. We already observed in § 5 that the functions a_i are quotients of the form

$$a_i = P_i\, \tau / \tau$$

where P_i is a polynomial differential operator in $\{\partial/\partial t_k\}$; indeed, P_i is the coefficient of z^{-i} in the formal expansion of the expression

$$\exp\left[-\sum_1^\infty \frac{1}{r} z^{-r}(\partial/\partial t_r)\right].$$

It follows at once from this that the operator P_i lowers weight by i (where, as always, t_k has weight k). Thus in the power series expansion of the numerator $P_i\,\tau$ in the expression for a_i, only terms of weight at least $\ell - i$ can occur. (If $\ell - i < 0$, this statement is vacuous.) Hence when we put $t_2 = t_3 = \ldots = 0$ in the numerator, the lowest power of t_1 that can occur is $t_1^{\ell - i}$ (any terms involving a lower power of t_1 must also involve a higher t_k, and hence vanish when we set $t_2 = \ldots = 0$). Proposition (5.17) follows at once from this and (8.6). In fact the argument shows also that the order of the pole of any a_i cannot be more than ℓ (*).

(*) Added in proof. According to G. Laumon (private communication) the order cannot be more than $-s_0$.

9. The τ-function and the theta function

Let X be a compact Riemann surface of genus g, and let J be the Jacobian of X: it is the identity component of the group $H^1(X, \mathcal{O}^\times)$, where $\mathcal{O}$ is the sheaf of holomorphic functions on X. We set $U = H^1(X, \mathcal{O})$ and $\Lambda = H^1(X, \mathbf{Z})$. The map $f \mapsto e^f$ induces a sheaf homomorphism $\mathcal{O} \to \mathcal{O}^\times$ with kernel $2\pi i\mathbf{Z}$, from which we get the exact sequence

$$0 \to \Lambda \to U \to J \to 0$$

(the kernel is really $H^1(X, 2\pi i\mathbf{Z})$, but we identify this with $H^1(X, \mathbf{Z})$ in the obvious way). We recall that U is a g-dimensional complex vector space, Λ is a lattice in U, and $J = U/\Lambda$ is a complex torus.

We denote by $B : \overline{U} \times U \to \mathbf{C}$ the unique Hermitian form whose imaginary part is the $\mathbf{R}$-bilinear extension of the intersection pairing $\Lambda \times \Lambda \to \mathbf{Z}$. We fix a quadratic form $q : \Lambda \to \mathbf{Z}/2\mathbf{Z}$ such that

$$q(\lambda + \mu) - q(\lambda) - q(\mu) = \lambda . \mu \pmod 2,$$

where $\lambda . \mu$ is the intersection pairing. Then the *theta function* of X (see, for example, [15]) is the holomorphic function $\theta : U \to \mathbf{C}$ defined by

$$\theta(u) = \sum_{\lambda \in \Lambda} (-1)^{q(\lambda)} e^{-\frac{1}{2}\pi B(\lambda, \lambda + 2u)}.$$

It is characterized (up to a constant factor) by the functional equation

$$\theta(u + \lambda) = (-1)^{q(\lambda)} e^{\frac{1}{2}\pi B(\lambda, \lambda + 2u)} \theta(u) \tag{9.1}$$

(for $u \in U$, $\lambda \in \Lambda$). It follows at once that we have

$$\theta(u + \lambda) = C\theta(u)\, \theta(\lambda)\, e^{\pi B(\lambda, u)}$$

(where $C = \theta(0)^{-1}$). We shall use the fact that this relation too characterizes the theta function up to certain simple transformations. More precisely, we have the following.

Lemma **9.2**. — *Let* $\widetilde{\theta} : U \to \mathbf{C}$ *be a holomorphic function such that*

$$\widetilde{\theta}(u + \lambda) = \widetilde{C}\,\widetilde{\theta}(u)\, \widetilde{\theta}(\lambda)\, e^{\pi B(\lambda, u)}$$

for all $u \in U$, $\lambda \in \Lambda$, *and some (non-zero) constant* $\widetilde{C}$. *Then we can find a constant* A, *a* $\mathbf{C}$*-linear map* $\alpha : U \to \mathbf{C}$ *and a point* $\beta \in U$ *such that*

$$\widetilde{\theta}(u) = A e^{\alpha(u)} \theta(u - \beta).$$

Proof. — Set

$$G(u) = \frac{\widetilde{C}\,\widetilde{\theta}(u)}{C\theta(u)}.$$

Then $G(u + \lambda) = G(u)\, G(\lambda)$, and the restriction of G to Λ is a homomorphism $\Lambda \to \mathbf{C}^\times$. Choose an $\mathbf{R}$-linear map $\gamma : U \to \mathbf{C}$ such that $G(\lambda) = e^{\gamma(\lambda)}$ for $\lambda \in \Lambda$. Splitting γ

into its **C**-linear and **C**-antilinear parts and using the non-degeneracy of the form B, we see that there are α and β as in the statement of the lemma such that

$$G(\lambda) = e^{\alpha(\lambda) - \pi B(\lambda, \beta)}$$

for $\lambda \in \Lambda$. If we set $H(u) = e^{-\alpha(u)}\tilde{\theta}(u + \beta)/\theta(u)$, then $H(u + \lambda) = H(u)$ for all $\lambda \in \Lambda$; hence the holomorphic function

$$\theta_1(u) = e^{-\alpha(u)}\tilde{\theta}(u + \beta)$$

satisfies the same functional equation (9.1) as the theta function, and must therefore be a constant multiple of it. The lemma follows.

Remark **9.3.** — Obviously, the constant A is uniquely determined by $\tilde{\theta}$. The α and β are not quite uniquely determined, because the map γ occurring in the proof of the lemma is determined only up to addition of a map γ_0 with $\gamma_0(\Lambda) \subset 2\pi i\mathbf{Z}$. However, it is easy to check that this would change the corresponding β only by a lattice point, so the projection of β onto the Jacobian U/Λ is uniquely determined. Also, α is uniquely determined once we have chosen β.

The τ-function is a function on the group Γ_+; our next task is to explain how we can regard the theta function too as defined on Γ_+, so that it makes sense to compare the two functions. We fix a point $x_\infty \in X$ and a local parameter z as in § 6. We shall use z to identify $X_\infty \subset X$ with the disc $D_\infty = \{|z| \geq 1\}$ in the Riemann sphere. We denote by V the vector space of all holomorphic maps $f: D_0 \to \mathbf{C}$ with $f(0) = 0$. As in § 5, we identify V with Γ_+ via the map $f \mapsto e^f$, and we shall regard the τ-function as a function on V. Now, any $f \in V$ (indeed, any holomorphic function on S^1) can be regarded as a cocycle for the Cech cohomology group $H^1(X, \mathcal{U})$, where $\mathcal{U} = \{U_0, U_\infty\}$ is an open covering of X as in the proof of (6.1). Using again the fact that we can calculate the cohomology of X from any such covering, we get a surjective homomorphism

$$V \to H^1(X, \mathcal{O}) = U.$$

Thus if K_0 denotes the kernel of this map, we can regard the theta function as a K_0-invariant function on V. Now, K_0 is the linear subspace of V consisting of all functions $k \in V$ which can be written in the form $k = k_0 + k_\infty$, where k_0 and k_∞ are holomorphic functions on X_0 and X_∞, respectively; the splitting is unique if we normalize k_∞ so that $k_\infty(\infty) = 0$. We denote by $\tilde{V}$ the vector space of all such maps k_∞. Let K be the kernel of the composite map $V \to U \to J$; it consists of all functions $k \in V$ such that there is a factorization (necessarily unique)

$$e^k = \varphi_k e^{k_\infty} \tag{9.4}$$

where $k_\infty \in \tilde{V}$ and φ_k is a non-vanishing holomorphic function on X_0. Clearly $K/K_0 \cong \Lambda$, so that K_0 is indeed the identity component of K, as the notation suggests. In the proof of (9.10) below we shall give an explicit description of the integral cohomology class corresponding to an element $k \in K$.

We now fix a line bundle $\mathscr{L}$ of degree g over X and a trivialization φ as in § 6; let $W \in Gr$ be the corresponding space. For simplicity we assume that W is transverse and that the function $\tau = \tau_W : V \to \mathbf{C}$ is normalized as usual by $\tau(0) = 1$. The τ-function is not usually K_0-invariant: however, we show next that a simple modification of it is. We define a map $a : K \to \widetilde{V}$ by $a(k) = k_\infty$, where k_∞ is as in (9.4). Clearly a is a homomorphism, and its restriction to K_0 is a $\mathbf{C}$-linear map.

Lemma **9.5.** — *Let* $f \in V$, $k \in K$. *Then we have*

$$\tau(f+k) = \tau(f)\,\tau(k)\,e^{S(a(k),f)},$$

where S *is the multiplier relating the actions of* Γ_+ *and* Γ_- *on the bundle* Det* *(see* (3.6)*).*

Proof. — By the definition of the τ-function (see (3.2)), we have

$$\tau(f+k)\,e^{-f-k}\,\sigma(W) = \sigma(e^{-f-k}\,W). \tag{9.6}$$

From the definition of W, it is clear that $\varphi_k W = W$, so we have $e^{-k}\,W = e^{-a(k)}\,W$ for $k \in K$. Using this and the fact that σ is Γ_--equivariant (see (3.7)), we find

$$\tau(k)\,e^{-k}\,\sigma(W) = e^{-a(k)}\,\sigma(W). \tag{9.7}$$

The right hand side of (9.6) is equal to

$$e^{-a(k)}\,\sigma(e^{-f}\,W) = \tau(f)\,e^{-a(k)}\,e^{-f}\,\sigma(W) = \tau(f)\,e^{S(a(k),f)}\,e^{-f}\,e^{-a(k)}\,\sigma(W).$$

Inserting (9.7) into this and cancelling the non-zero vector $e^{-f-k}\,\sigma(W)$, we get the lemma.

If we apply (9.5) when both f and k belong to K, we find that

$$S(a(k),\ell) - S(a(\ell),k) \in 2\pi i\mathbf{Z}$$

for all k, $\ell \in K$. Extend a to an $\mathbf{R}$-linear map $V \to \widetilde{V}$; since K spans V over $\mathbf{R}$, the extension is unique, and we have

$$S(a(f),g) - S(a(g),f) \in i\mathbf{R} \tag{9.8}$$

for all f, $g \in V$. Write $a = b + c$, where b is $\mathbf{C}$-linear and c is antilinear. Then (9.8) implies that

$$S(b(f),g) = S(b(g),f),$$
$$S(c(f),g) = \overline{S(c(g),f)}$$

for all f, $g \in V$. Since $a \mid K_0$ is $\mathbf{C}$-linear, we have $c(K_0) = 0$; thus c, and hence also the Hermitian form $(f,g) \mapsto S(c(f),g)$, are well defined on $U = V/K_0$. Set $\tau_1(f) = \tau(f)\,e^{-\frac{1}{2}S(b(f),f)}$. Then from (9.5) we have

$$\tau_1(f+k) = \tau_1(f)\,\tau_1(k)\,e^{S(c(k),f)}.$$

In particular, the restriction of τ_1 to K_0 is a homomorphism $K_0 \to \mathbf{C}^\times$. Choose a $\mathbf{C}$-linear map $\eta : V \to \mathbf{C}$ such that $\tau_1(k) = e^{\eta(k)}$ when $k \in K_0$; set $\tau_2(f) = \tau_1(f)\, e^{-\eta(f)}$. Then $\tau_2(f+k) = \tau_2(f)$ for $k \in K_0$. Thus τ_2 is well defined on U, and it satisfies

$$(9.9) \qquad \tau_2(u+\lambda) = \tau_2(u)\,\tau_2(\lambda)\, e^{S(c(\lambda),u)}$$

for $\lambda \in \Lambda = K/K_0$. But now we have the following crucial result.

Proposition **9.10.** — *For all k, $\ell \in K$, we have*

$$S(c(k),\ell) - S(c(\ell),k) = 2\pi i[k].[\ell],$$

where $[k]$, $[\ell]$ denote the classes of k, ℓ in the group $K/K_0 = \Lambda = H^1(X, \mathbf{Z})$.

The proposition shows that the Hermitian form occurring in the exponent in (9.9) is π times the form B occurring in the definition of the theta function. We can therefore apply (9.2) to obtain the main result of this section.

Theorem **9.11.** — *The τ-function $\tau_W : V \to \mathbf{C}$ is related to the theta function by*

$$\tau_W(f) = A e^{\alpha_W(f) + \frac{1}{2}S(b(f),f)}\, \theta(\bar{f} - \beta_W),$$

where A is a constant, $\alpha_W : V \to \mathbf{C}$ is a linear map, β_W is a point of U, and $\bar{f}$ denotes the projection of f onto $U = V/K_0$.

Remarks

(i) Note that the quadratic term $\frac{1}{2} S(b(f),f)$ depends only on X and z.

(ii) By (9.3), the projection of β_W onto the Jacobian J is uniquely determined by W. If W moves according to one of the KP flows, then β_W moves along the corresponding straight line in J.

(iii) There seems no point in trying to be more explicit about the map α, since it depends on the choice of trivialization φ (see (3.8)).

It remains to give the proof of (9.10). For this we fix a basis $\Delta = \{\alpha_i, \beta_i\}$, $1 \leq i \leq g$, for $H_1(X, \mathbf{Z})$ of the standard kind, that is, such that $\alpha_i.\beta_i = 1$ and all other intersections are zero. We can then regard the Riemann surface X in the classical way as a quotient of a polygon Y with $4g$ edges arranged in groups of four $(\alpha_i, \beta_i, \alpha_i^{-1}, \beta_i^{-1})$ (we get X from Y by identifying the two edges corresponding to each element of Δ). We suppose Y chosen so that the disc X_∞ in X corresponds to a small disc Y_∞ in the interior of Y; let Y_0 be the complement of the interior of Y_∞. If $k \in K$, then $k = k_0 + k_\infty$, where k_0 and k_∞ are functions on Y_0 and Y_∞, respectively. Now, $e^{k_0} = \varphi_k$ is a function on X: that means that the values of k_0 at corresponding points of the two edges of Y corresponding to a generator $\gamma \in \Delta$ differ by an integer multiple of $2\pi i$, say by $2\pi i n(k, \gamma)$. The cohomology class defined by k is then given by

$$[k] = \sum_{\gamma \in \Delta} n(k, \gamma)\, \gamma^*$$

where $\{\gamma^*\}$ is the basis of $\mathrm{H}^1(\mathrm{X}, \mathbf{Z}) = \Lambda$ dual to Δ.

Now, we have

$$\begin{aligned} \mathrm{S}(c(k), \ell) - \mathrm{S}(c(\ell), k) &= \mathrm{S}(a(k), \ell) - \mathrm{S}(a(\ell), k) \\ &= \frac{1}{2\pi i} \int_{\mathrm{S}^1} (\ell k'_\infty - k \ell'_\infty). \end{aligned}$$

After a short calculation we find that this is equal to

$$\frac{1}{2\pi i} \int_{\mathrm{S}^1} k_0 \ell'_0.$$

Since k_0 and ℓ_0 are holomorphic functions on Y_0, we can replace S^1 by the boundary of Y in this integral. The contribution to the integral of a typical set of four edges

$$\circ \xrightarrow{\ \alpha_i\ } \circ \xrightarrow{\ \beta_i\ } \circ \xrightarrow{\ \alpha_i^{-1}\ } \circ \xrightarrow{\ \beta_i^{-1}\ } \circ$$

can be reduced to an integral over the middle pair (β_i, α_i^{-1}): we obtain

$$n(k, \beta_i) \int_{\beta_i} \ell'_0 + n(k, \alpha_i) \int_{\alpha_i} \ell'_0 = 2\pi i \{ - n(k, \beta_i)\, n(\ell, \alpha_i) + n(k, \alpha_i)\, n(\ell, \beta_i) \}.$$

Summing over i and using the fact that the intersection matrix of the basis $\{\alpha_i^*, \beta_i^*\}$ is the same as that of $\{\alpha_i, \beta_i\}$, we see that the integral is indeed equal to $2\pi i [k] . [\ell]$.

The Baker function and the theta function

If we combine (9.11) with (5.14), we obtain a formula expressing the Baker function (of a space W arising from a Riemann surface) in terms of the theta function. As we mentioned in the introduction, such a formula is well known in the Russian literature (see, for example, [10, 11, 36]). However, it is perhaps not immediately obvious that the Japanese formula (5.14) coincides with the Russian one: so at the suggestion of the referee we end this section by offering a fairly detailed comparison of the two formulas.

The Russian formula is expressed in terms of the classical Riemann theta function, whose definition involves a choice of canonical homology basis $\{\alpha_i, \beta_i\}$ as in the proof of (9.10) above: we suppose such a basis fixed from now on. The classical theta function is a function on the dual space R^* of the space R of global holomorphic differentials on X; but R^* is usually identified with $\mathbf{C}^g$ via the basis

$$\left\{ \omega \mapsto \int_{\alpha_i} \omega \right\}, \quad \omega \in \mathrm{R}.$$

On the other hand we have the natural pairing

$$H^1(X, \mathcal{O}) \otimes H^0(X, \Omega) \to H^1(X, \Omega) \cong \mathbf{C},$$

where Ω is the sheaf of holomorphic differentials on X, which canonically identifies R^* with the space $U = H^1(X, \mathcal{O})$ on which our theta function was defined. In what follows we shall use without further comment these identifications $U \cong R^* \cong \mathbf{C}^g$. The choice of homology basis $\{\alpha_i, \beta_i\}$ gives a natural choice for the form $q : \Lambda \to \mathbf{Z}/2\mathbf{Z}$ occurring in our version of the theta function, namely, we can choose q to vanish on the basis for $\Lambda \cong H^1(X, \mathbf{Z})$ dual to $\{\alpha_i, \beta_i\}$. It is then easy to check that our theta function differs from the classical one only by a factor $\exp Q(u, u)$, where Q is a symmetric $\mathbf{R}$-bilinear form on U. Thus if we use the classical theta function, theorem 9.11 remains true except that the quadratic form is different. From now on we write θ for the classical theta function.

With these preliminaries, we can now explain the Russian formula relating the Baker function and the theta function. We follow the account given in [36], to which we refer the reader for more details. With Krichever, we fix a non-special positive divisor $\mathscr{D} = \{P_1, \ldots, P_g\}$ on X; without loss of generality (see (6.5)) we suppose the points P_i lie outside the disc $D_\infty \subset X$. We want to write down the Baker function ψ_W, where W is the closure of the space of analytic functions on S^1 which extend to meromorphic functions on X_0 that are regular except for (possible) simple poles at the points P_i. We fix a base point $P_0 \neq x_\infty$ in X, and let $A : X \to R^* \cong \mathbf{C}^g$ be the corresponding Abel map, given by

$$A(P)(\omega) = \int_{P_0}^{P} \omega \quad (P \in X, \quad \omega \in R).$$

The map A is well defined only modulo the period lattice Λ (because of the choice of path of integration). Let $C \in \mathbf{C}^g$ be a constant vector such that the function (on X) $\theta(A(P) - C)$ vanishes precisely when $P = P_1, \ldots, P_g$. For $n = 1, 2, \ldots$, let ω_n be the differential of the second kind which has zero α-periods and is regular except for a singularity at x_∞ with principal part $d(z^n)$. Let $W_n \in \mathbf{C}^g$ be the vector of β-periods of ω_n. Consider the expression

$$\textbf{(9.12)} \qquad \exp\left\{ \sum_i t_i \int_{P_0}^{P} \omega_i \right\} \frac{\theta(A(P) + \Sigma t_i W_i - C)}{\theta(A(P) - C)}.$$

It is understood that the path of integration in the first term is the same as that used in the Abel map; it is then easy to check (see [36], ch. 3, § 1) that (9.12) is a well defined function of $P \in X$, although the individual terms in it are not. It is obvious that when restricted to $S^1 \subset X$ the function (9.12) belongs to W for each fixed $\mathbf{t}$, and has the form

$$\exp \Sigma t_i z^i (a_0(\mathbf{t}) + a_1(\mathbf{t}) z^{-1} + \ldots).$$

Thus to get the Baker function ψ_W, we have only to divide by $a_0(\mathbf{t})$. That yields the final formula

$$\textbf{(9.13)} \qquad \psi_W = \exp\left\{\Sigma t_i \int_{P_0}^{P} \omega_i\right\} \exp\{-\Sigma t_i b_{i0}\} \frac{\theta(A(P) + \Sigma t_i W_i - C)\,\theta(A(x_\infty) - C)}{\theta(A(P) - C)\,\theta(A(x_\infty) + \Sigma t_i W_i - C)}$$

where the constants b_{i0} are defined from the expansions

$$\int_{P_0}^{z} \omega_n = z^n + \sum_{0}^{\infty} b_{nr} z^{-r}$$

for z near x_∞.

The formula (9.13) is global (P can run over the whole Riemann surface X). We now restrict it to $P \in D_\infty$ and accordingly write z instead of P. We claim that (9.13) can then be identified with the formula obtained by substituting (9.11) into (5.14). Note first that the quotient

$$\theta(A(x_\infty) - C)/\theta(A(z) - C)$$

in (9.13) is nothing but a function of the form $1 + c_1 z^{-1} + \dots$; it comes from the uninteresting linear term α in (9.11). The exponential terms in (9.13) can be written

$$\exp\{\Sigma t_i z^i\} \exp\left\{\sum_{i,j=1}^{\infty} t_i b_{ij} z^{-j}\right\};$$

the second factor here is the contribution to (9.13) coming from the quadratic term in (9.11). To complete our check that (5.14) and (9.13) agree, we have still to see two things: (i) that the vectors $W_i \in \mathbf{C}^g$ corresponding to the different t_i agree with those in (9.11) (obtained by regarding the functions z^i as cocycles for $H^1(X, \mathcal{O})$); (ii) that the difference in the arguments of the two remaining theta function terms in (9.13) agrees with the q_ζ in (5.14). For (i), we use the fact that the canonical pairing $U \times R \to \mathbf{C}$ can be derived from the pairing $V \times R \to \mathbf{C}$ given by

$$(f, \omega) \mapsto \frac{1}{2\pi i}\int_{S^1} f\omega;$$

the desired assertion then reduces to something well known (see, for example, [36], (2.1.21)). Concerning (ii), note that the difference in question is

$$A(z) - A(x_\infty) = A_\infty(z),$$

where A_∞ is the Abel map defined using the base point x_∞. Hence the result we need is the following.

Lemma **9.14.** — *Let $\Gamma_+ \to U \to J = U/\Lambda$ be the map used earlier in this section (defined by regarding an element of Γ_+ as a transition function for a line bundle on X). Then for $|\zeta| > 1$, the image of q_ζ under this map is $A_\infty(\zeta)$.*

Proof. — We write q_ζ in the form

$$q_\zeta(z) = (z^{-1} - \zeta^{-1})\, z.$$

The two factors here are transition functions for the line bundles corresponding to the divisors $[\zeta]$ and $[-x_\infty]$, respectively. Thus the image of q_ζ in the Jacobian is $[\zeta] - [x_\infty]$, which is indeed $A_\infty(\zeta)$.

Finally, we point out that one can reverse some of the arguments we have just given and *prove* (9.11) by comparing the formulas (5.14) and (9.13). This argument is indicated in [5], and is indeed the only possible one there, because at this point in [5] the τ-function is *defined* in terms of the Baker function by the formula (5.14). In our context, however, we have an independent definition of the τ-function, so it seemed to us very desirable to give a direct proof of (9.11), avoiding the use of the Baker function.

10. Appendix: the representation theory of the loop group

In this paper we have not mentioned the representation theory of the loop group $LGL_n(\mathbf{C})$, whereas the Japanese papers [5] put it in the foreground. The difference, however, is more apparent than real, as we shall now explain. We shall begin by describing the situation without any attempt at justification, and at the end we shall return to give some indications about the proofs.

It will be convenient in this section to let Gr denote the "Hilbert-Schmidt" Grassmannian of H, consisting of closed subspaces W of H such that the projection $W \to H_+$ is Fredholm and the projection $W \to H_-$ is Hilbert-Schmidt. Alternatively, Gr consists of the graphs of all Hilbert-Schmidt operators $H_S \to H_S^\perp$. It is clearly a Hilbert manifold. We shall write $LGL_n(\mathbf{C})$ for the group of *smooth* loops.

We have seen* that a central extension of $LGL_n(\mathbf{C})$ by $\mathbf{C}^\times$ acts on the holomorphic line bundle Det^* on Gr. This means that $LGL_n(\mathbf{C})$ acts projectively on the space $\Gamma(\mathrm{Det}^*)$ of all holomorphic sections of Det^*. With the topology of uniform convergence on compact sets, $\Gamma(\mathrm{Det}^*)$ is a complete topological vector space. It is the so-called "basic" irreducible projective representation of $LGL_n(\mathbf{C})$.

For any indexing set $S \in \mathscr{S}$ the "Plücker coordinate" $W \mapsto w^S$ (introduced in § 8) is an element of $\Gamma(\mathrm{Det}^*)$. We shall denote it by π_S. In fact the π_S span a dense subspace; and there is a natural Hilbert space $\mathscr{H}$ inside $\Gamma(\mathrm{Det}^*)$—it can be thought of as the "square-integrable" holomorphic sections—for which the π_S form an orthonormal basis. The subgroup LU_n of $LGL_n(\mathbf{C})$ acts by a projective unitary representation on $\mathscr{H}$.

The geometrical significance of $\mathscr{H}$ is that there is a natural antiholomorphic embedding

$$\Omega : \mathrm{Gr} \to P(\mathscr{H})$$

* Strictly speaking, in § 3 we considered only one component of $LGL_n(\mathbf{C})$ and Gr; but it is not hard to extend the discussion to include the other components (see [17]).

of the infinite dimensional complex manifold Gr in the projective space of $\mathscr{H}$. It assigns to $W \in \mathrm{Gr}$ the ray in $\mathscr{H}$ containing the section Ω_w of Det* defined by

$$\Omega_w(w') = \det \langle w, w' \rangle,$$

where w is an admissible basis of W. (Here $\langle w, w' \rangle$ denotes the matrix whose (i, j)-th element is $\langle w_i, w'_j \rangle$; and we are thinking of a section of Det* as a $\mathscr{E}$-equivariant map $\mathscr{P} \to \mathbf{C}$.) The embedding Ω is equivariant with respect to LU_n.

The vector in $\mathscr{H}$ corresponding to H_+ with its standard basis, i.e. the canonical section σ of Det* (cf. § 3), is called the *vacuum vector* Ω_0.

If we think of the representation $\mathscr{H}$ as given, rather than the manifold Gr, then the discussion of §§ 3 and 5 can be very simply translated. The crucial formula is the definition of the τ-function, which becomes

$$\tau_W(g) = \langle \Omega_0, g\Omega_w \rangle,$$

where $g \in \Gamma_+$, and w is an admissible basis for W. This is the definition in parts of [5], except that these authors appear to have in mind only the group of polynomial loops, corresponding to our Grassmannian Gr_0.

Two other realizations of the Hilbert space $\mathscr{H}$ are of importance. To describe the first, notice that the connected components of Gr are indexed by the integers, and that correspondingly

$$\mathscr{H} = \bigoplus_{k \in \mathbf{Z}} \mathscr{H}_k,$$

where $\mathscr{H}_k$ consists of functions on the k-th connected component. We saw in § 2 that the group Γ_- of holomorphic functions in the disc $|z| \geq 1$ acts freely on Gr. Let X denote the orbit of H_+ under Γ_-. The restriction of Det* to X is canonically trivial; so holomorphic sections of Det* restrict to give complex-valued holomorphic functions on X. Writing a general element of Γ_- in the form

$$1 + h_1 z^{-1} + h_2 z^{-2} + \dots,$$

we think of functions on X as functions of the infinite sequence of complex variables $h_1, h_2, h_3, \dots$. In fact sections of Det* over the component of Gr containing H_+ are determined by their restrictions to X, and we have.

Proposition **10.1.**

(i) *If* $S \in \mathscr{S}$ *has virtual cardinal zero then the Plücker coordinate* $\pi_S \in \Gamma(\mathrm{Det}^*)$ *restricts to the Schur function* $F_S(h_1, h_2, \dots)$. *(Cf. § 8.)*

(ii) $\mathscr{H}_0$ *can be identified with the completion of the ring of symmetric polynomials* $\mathbf{Z}[h_1, h_2, \dots]$ *with respect to its standard inner product* [13]; *equivalently, it is the space of* L^2 *holomorphic functions on* Γ_- *with respect to the Gaussian measure*

$$d\mu(g) = e^{-\Sigma n|a_n|^2} \prod da_n \, d\bar{a}_n,$$

where $g = \exp \Sigma a_n z^{-n}$.

The second concrete realization of $\mathscr{H}$ is as the exterior algebra on the Hilbert space $H_+ \oplus \overline{H}_-$. As H_+ and $\overline{H}_-$ have the orthonormal bases $\{z^k\}_{k\geq 0}$ and $\{\bar{z}^k\}_{k<0}$ respectively, the exterior algebra $\Lambda(H_+ \oplus \overline{H}_-)$ has an orthonormal basis of the form

(10.2) $$\bar{z}^{a_1} \wedge \ldots \wedge \bar{z}^{a_k} \wedge z^{b_1} \wedge \ldots \wedge z^{b_m},$$

where $a_1 < \ldots < a_k < 0 \leq b_1 < \ldots < b_m$. These basis elements correspond exactly to the indexing sets $S \in \mathscr{S}$ with which we are familiar: we write $S \setminus \mathbf{N} = \{a_1, \ldots, a_k\}$ and $\mathbf{N} \setminus S = \{b_1, \ldots, b_m\}$. Thus we can denote the element (10.2) by z^S; the isomorphism $\Lambda(H_+ \oplus \overline{H}_-) \cong \mathscr{H}$ makes z^S correspond to the Plücker coordinate π_S.

A more interesting and also more relevant way of constructing the map $\Lambda(H_+ \oplus \overline{H}_-) \to \mathscr{H}$ is by defining "fermionic field operators" on $\mathscr{H}$. These amount to an operator-valued distribution $\theta \mapsto \varphi(\theta)$ on the circle, satisfying the anticommutation relations

$$[\varphi(\theta_1), \varphi(\theta_2)]_+ = 0.$$

$$[\varphi(\theta_1), \varphi(\theta_2)^*]_+ = \delta(\theta_1 - \theta_2).$$

Then the map $\Lambda(H_+ \oplus \overline{H}_-) \to \mathscr{H}$ is

$$f_1 \wedge \ldots \wedge f_k \wedge \bar{g}_1 \wedge \ldots \wedge \bar{g}_m \mapsto \varphi_{f_1} \ldots \varphi_{f_k} \varphi^*_{g_1} \ldots \varphi^*_{g_m} \Omega_0,$$

where $$\varphi_f = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, \varphi(\theta)\, d\theta.$$

The highly singular "vertex operator" $\varphi(\theta)$ is constructed from the action of $\Gamma = \mathbf{L}\mathbf{C}^\times$ on $\mathscr{H}$ as the limit $\rho \to 1+$ of the action of $p_\zeta q_\zeta$, where $\zeta = \rho e^{i\theta}$, and

$$q_\zeta = 1 - \zeta^{-1} z \in \Gamma_+,$$

$$p_\zeta = (1 - \bar{\zeta}^{-1} z^{-1})^{-1} \in \Gamma_-.$$

The important formula (5.15) for the unique element of $W \cap (1 + H_-)$ can be written

$$\psi_W(0, e^{i\theta}) = \langle \Omega_0, \varphi(\theta)\, \Omega_w \rangle.$$

This is equivalent to (5.15), because

$$\begin{aligned}\langle \Omega_0, \varphi(\theta)\, \Omega_w \rangle &= \lim \langle \Omega_0, p_\zeta\, q_\zeta\, \Omega_w \rangle \\ &= \lim \langle p_\zeta^* \Omega_0, q_\zeta\, \Omega_w \rangle \\ &= \lim \langle \Omega_0, q_\zeta\, \Omega_w \rangle \\ &= \lim \tau_W(q_\zeta).\end{aligned}$$

Remarks about the proofs

Let $H_{m,n} = z^m H_+ / z^n H_+$ when $m \leq n$. Then Gr contains the finite dimensional Grassmannians $Y_n = \mathrm{Gr}(H_{-n,n})$, and the union of the Y_n is dense. The bundle Det^*

on Gr restricts to the usual Det^* on Y_n; so we know that $\Gamma(\mathrm{Det}^* \mid Y_n)$ can be identified with the exterior algebra $\Lambda(H_{-n,n})$. A section of Det^* is determined by its restrictions to the Y_n. Thus we have an inclusion

$$\text{(10.3)} \qquad \Gamma(\mathrm{Det}^*) \hookrightarrow \varprojlim_n \Lambda(H_{-n,n}).$$

Now $\Lambda(H_{-n,n})$ has a basis indexed by the 2^{2n} sets $S \in \mathscr{S}$ such that

$$[n, \infty) \subset S \subset [-n, \infty).$$

These come from the corresponding Plücker coordinates π_S in $\Gamma(\mathrm{Det}^*)$. This shows that the map (10.3) has a dense image, and also that the π_S span a dense subspace of $\Gamma(\mathrm{Det}^*)$.

To construct the Hilbert space $\mathscr{H}$ we begin by observing that $w \mapsto \Omega_w$ defines an antiholomorphic map

$$\Omega : \mathrm{Det} \to \Gamma(\mathrm{Det}^*)$$

which is antilinear on each fibre of Det. (Notice that if $w = \{z^s\}_{s \in S}$ then Ω_w is the Plücker coordinate π_S.) By transposing Ω we obtain a **C**-linear map

$$\Omega^* : F \to \Gamma(\mathrm{Det}^*),$$

where F is the antidual of $\Gamma(\mathrm{Det}^*)$, i.e. the space of continuous antilinear maps $\Gamma(\mathrm{Det}^*) \to \mathbf{C}$. This gives us a hermitian form $\bar{F} \times F \to \mathbf{C}$ defined by

$$(\alpha, \beta) \mapsto \alpha(\Omega^* \beta).$$

In fact Ω^* is injective and has dense image, because the Ω_w span $\Gamma(\mathrm{Det}^*)$, and the Hilbert space completion $\mathscr{H}$ of F is sandwiched between F and $\Gamma(\mathrm{Det}^*)$. It is clear that the π_S form an orthonormal basis of $\mathscr{H}$. Because Ω is equivariant with respect to LU_n (or, more accurately, with respect to a central extension of LU_n by the circle), it follows that LU_n acts unitarily on $\mathscr{H}$.

The proof of (10.1) (i) is almost exactly the same as that of (8.2); the second part is then routine.

For a discussion of vertex operators we refer to [17] or [18].

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