A FORMULA FOR EULER CHARACTERISTICS OF TAUTOLOGICAL LINE BUNDLES ON THE DELIGNE-MUMFORD MODULI SPACES

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Abstract. We compute holomorphic Euler characteristics of the line bundles $\bigotimes_{i=1}^n L_i^{\otimes d_i}$ over the moduli space $\overline{M}_{0,n}$ of stable n-pointed curves of genus 0, where L_i is the holomorphic line bundle over $\overline{M}_{0,n}$ formed by the cotangent lines at the i-th marked point.

§1 Introduction

Let $M_{0,n}$ be the space of n (ordered) distinct points $(x_1, \cdots, x_n)$ on $\mathbf{P}^1$ modulo $\mathbf{P}SL_2(\mathbf{C})$. There is a natural compatification of this space by adding n-pointed singular stable curves to the boundary. The compactified space $\overline{M}_{0,n}$ is called Deligne-Mumford moduli space. Each point x_i induces a line bundle on $M_{0,n}$ with fibre $T^*_{x_i}\mathbf{P}^1$. This line bundle can be extended to $\overline{M}_{0,n}$, which we denote by L_i. (For rigorous definitions see the next section.)

Denote $\chi_{d_1,\cdots,d_n}$ the holomorphic Euler characteristics of the line bundles $\bigotimes_{i=1}^n L_i^{\otimes d_i}$

$$\chi_{d_1,\cdots,d_n} = \sum_k (-1)^k dim_{\mathbf{C}} H^k(\overline{M}_{0,n}, \bigotimes_{i=1}^n L_i^{\otimes d_i}). \tag{1}$$

Let q_i's be formal variables. Introduce the generating function

$$\begin{aligned} G(q_1, \cdots, q_n) &= \sum_{(d_1,\cdots,d_n)} \chi_{d_1,\cdots,d_n} q_1^{d_1} \cdots q_n^{d_n} \\ &= \chi(\bigotimes_{i=1}^n \frac{1}{1 - q_i L_i}). \end{aligned} \tag{2}$$

Our main result is:

Theorem

$$G(q_1, \cdots, q_n) = (1 + \sum_{i=1}^n \frac{q_i}{1-q_i})^{n-3} \prod_{i=1}^n \frac{1}{1-q_i}. \tag{3}$$

In fact, several examples show that this formula might actually give the dimensions of the spaces of holomorphic sections of the line bundles, i.e. all higher cohomology groups vanish. We will indicate some of them in §3.

Here we have several remarks. First, this formula is related to the theory of two-dimensional gravity. Recall that the tree level correlation functions of two-dimensional gravity are defined to be [DW]:

$$N_{d_1,\cdots,d_n} =< \prod_{i=1}^{n} c_1(L_i)^{d_i}, \overline{M}_{0,n} > . \tag{4}$$

Let us introduce another formal variable z_i, which is related to q_i by the formula $q_i = e^{z_i}$. If we define the generating function as

$$\begin{aligned} F(z_1,\cdots,z_n) &= \sum_{(d_1,\cdots,d_n)\geq 0} N_{d_1,\cdots,d_n} z_1^{d_1}\cdots z_n^{d_n} \\ &= \int_{\overline{M}_{0,n}} \frac{1}{(1-z_1c_1(L_1))\cdots(1-z_nc_1(L_n))}, \end{aligned} \tag{5}$$

then using the string equation [DW] one easily gets

$$F(z_1,\cdots,z_n) = (z_1+\cdots+z_n)^{n-3}. \tag{6}$$

Thus the Euler characteristics $\chi_{d_1,\cdots,d_n}$ can be regarded as the correlation functions in the corresponding K-theory and formula (6) is a limiting case of our theorem in the cohomology theory.

We remark also that our "correlation functions" can be written as:

$$\chi_{d_1,\cdots,d_n} = \int_{\overline{M}_{0,n}} ch(\bigotimes L_i^{\otimes d_i})Td(\overline{M}_{0,n}) \tag{7}$$

by the Riemann-Roch Formula. Since the structure of the Todd classes of $\overline{M}_{0,n}$ is still not well understood at this moment, our formula may provide some information.

The idea of using generating function was suggested by A. Givental and motivated by the fixed point localization technique [G] [Ko] in the equivariant quantum cohomology theory. Notice that the formula (2) can be written as

$$\int_{\overline{M}_{0,n}} \frac{Td(\overline{M}_{0,n})}{\prod_i 1-e^{c_1(L_i)+z_i}},$$

and the denominator of the intergrand is exactly that of the holomorphic Bott-Lefschetz fixed point formula, with $c_1 + z_i$ interpreted as the equivariant first Chern class (of the fibrewise $U(1)$-action).

In the next section we will review some basic facts on stable curves and its moduli spaces. The proof of the formula (3) will be given in §3.

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§2 The stable curves and their moduli spaces

Let $\mathbf{C}$ be the field of complex numbers, over which all schemes are defined.

An n-pointed stable curve $(C; x_1, \cdots, x_n)$ of genus zero is a connected and reduced curve of arithmetic genus zero with at most ordinary double points such that C is smooth at x_i, $x_i \neq x_j$ (for $i \neq j$) and every component has at least 3 special points (marked points + singular points). ([Kn]). A family of n-pointed genus zero stable curves over an algebraic scheme S is a flat, projective morphism $\pi : F \to S$ with n sections $x_1, \cdots, x_n$ such that each geometric fibre $(F_s; x_1(s), \cdots, x_n(s))$ is an n-pointed stable curve of genus zero. Two families $(\pi : F \to S; x_1, \cdots, x_n)$ and $(\pi' : F' \to S; x_1{}', \cdots, x_n{}')$ are isomorphic if there exists an isomorphism $h : F \to F'$ over S such that $h \circ x_i = x_i{}'$.

Define the moduli functor $\overline{\mathcal{M}}_{0,n}$ to be a contravariant functor from the category of algebraic schemes to the category of sets, which assigns to each algebraic scheme S a set $\overline{\mathcal{M}}_{0,n}(S)$ of isomorphism classes of families of stable curves over S. In [Kn] Knudsen showed that there exists a fine moduli space $\overline{M}_{0,n}$ representing the functor $\overline{\mathcal{M}}_{0,n}$. Furthermore $\overline{M}_{0,n}$ is a smooth complete variety equipped with a universal curve $\pi : U_n \to \overline{M}_{0,n}$ and universal sections $x_1, \cdots, x_n$ (marked points).

In addition to representing $\overline{\mathcal{M}}_{0,n}$, $\overline{M}_{0,n}$ also gives an interesting compactification of the space of n distinct points on $\mathbf{P}^1$ modulo automorphisms of $\mathbf{P}^1$. This (noncompact) space is contained in $\overline{M}_{0,n}$ as an open subset, which is sometimes called the finite part of $\overline{M}_{0,n}$.

Knudsen also showed that U_n is isomorphic to $\overline{M}_{0,n+1}$. In particular, there are $n+1$ canonical morphisms:

$$\pi_{n+1}^{(i)} : \overline{M}_{0,n+1} \to \overline{M}_{0,n}, \qquad i = 1, \cdots n+1,$$

which map an $(n+1)$-pointed curve (as a point in $\overline{M}_{0,n+1}$) to an n-pointed curve by forgetting the i-th point. They are called forgetful maps. Notice that it might be necessary to contract unstable components (stabilization) when forgetting the marked points. In the following discusion only $\pi_{n+1}^{(n+1)}$ will be needed. Therefore we will denote it simply by π_{n+1} and call it *the* forgetful map .

The bundle L_i is defined as the conormal bundle to the universal section $x_i : \overline{M}_{0,n} \hookrightarrow U_n$. L_i is a holomorphic line bundle because the marked points are always nonsingular. To compare the difference of $L_i^{(n)}$ and $\pi_n^*(L_i^{(n-1)})$ ($L_i^{(n-1)}$ is the corresponding line bundle on $\overline{M}_{0,n-1}$) we will need some important divisors on $\overline{M}_{0,n}$.

Define D_i to be the divisor on $\overline{M}_{0,n}$ whose generic elements are the curves with two components, with $\{x_i, x_n\}$ on one branch and the remaining marked points on the other. It is known ([Kn]) that $\{D_i\}$ form a family of smooth divisors on $\overline{M}_{0,n}$ with normal crossings.

With these definitions we are able to state:

Lemma 1.

$$L_i^{(n)} \cong \pi_n^*(L_i^{(n-1)}) \otimes \mathcal{O}(D_i). \tag{8}$$

Proof. See [W]. A local holomorphic section of the locally free sheaf $L_i^{(n-1)}$ is represented by a local holomorphic one form ω on the curve $[C] \in \overline{M}_{0,n-1}$ evaluated at x_i and $\pi_n^*(\omega(x_i))$ vanishes exactly on the divisor D_i.

Q. E. D.

By definition a stable curve C is locally a complete intersection. Then the general theory of duality [H] will guarantee the existence of a canonical invertible sheaf K over C. In fact, let $f : C' \to C$ be the normalization of C, $y_1, \cdots, y_l; z_1, \cdots, z_l$ the points on C' such that $f(y_i) = f(z_i) = p_i$, $i = 1, \cdots, l$, are the double points of C. Then the canonical sheaf is the sheaf of one forms ω on C', regular except for simple poles at y's and z's and the $Res_{y_i}(\omega) = -Res_{z_i}(\omega)$([DM]). Therefore the Serre duality reads:

$$H^1(C, \mathcal{O}(D))^* = H^0(C, K \otimes \mathcal{O}(-D)) \tag{9}$$

for any divisor D on C. With (9) it is easy to see

Lemma 2. *Let $D = \sum_i d_i D_i$, $d_i \geq 0$, be a divisor on $\overline{M}_{0,n}$. We have*

$$H^1(C, \mathcal{O}(D)) = 0. \tag{10}$$

Proof. $H^1(C, \mathcal{O}_C) = 0$ by the definition of arithmetic genus. By Serre duality $H^1(C, \mathcal{O}_C) = H^0(C, K)$ and $H^1(C, \mathcal{O}(D))^* = H^0(C, K \otimes \mathcal{O}(-D)) = 0$ because $-D \leq 0$.

Q. E. D.

§3 The proof of theorem

We will prove the formula (3) by induction on n. First notice that the case $n = 3$ is trivial. Since $\overline{M}_{0,3}$ is a point, $\chi_{d_1 d_2 d_3} = 1$ for every triplet (d_1, d_2, d_3). By the definition of (2), $G(q_1, q_2, q_3) = \sum_{(d_1,d_2,d_3)\geq 0} q_1^{d_1} q_2^{d_2} q_3^{d_3}$. This is exactly the RHS of formula (3) as a formal power series. For the case of general n we need the following two reduction propositions.

Proposition 1. *Let $\pi_n : \overline{M}_{0,n} \to \overline{M}_{0,n-1}$ be the forgetful map. Denote $L_i^{(n)}$ and $L_i^{(n-1)}$ by L_i and l_i respectively, then*

$$ch(\pi_{n*}(\bigotimes_{i=1}^{n-1} \frac{1}{1 - q_i L_i})) = (1 + \sum_{i=1}^{n-1} \frac{q_i}{1 - q_i}) ch(\bigotimes_{i=1}^{n-1} \frac{1}{1 - q_i l_i}). \tag{11}$$

Here $\pi_{n*}(\alpha) = R^0\pi_{n*}(\alpha) - R^1\pi_{n*}(\alpha)$ are the Grothendieck's higher direct images for any element $\alpha \in K(\overline{M}_{0,n})$ and the LHS of equation (11) is a formal power series

$$\sum_{(d_1,\cdots,d_{n-1})} ch(\pi_{n*}(\bigotimes_{i=1}^{n-1} L_i^{\otimes d_i})) q_1^{d_1} \cdots q_{n-1}^{d_{n-1}}.$$

Proof. Let $\alpha := \bigotimes_{i=1}^{n-1} L_i^{\otimes d_i}$. Our problem is essentially to compute $R^i\pi_{n*}(\alpha)$. As stated in §2 $\pi_n : \overline{M}_{0,n} \to \overline{M}_{0,n-1}$ is in fact the universal curve, the fibre C_x over $x \in \overline{M}_{0,n-1}$ is just an $(n-1)$-pointed curve of genus 0. By lemma 1 $L_i = {\pi_n}^*(l_i) \bigotimes \mathcal{O}(D_i)$. Since

$\pi_n^*(l_i)$ is trivial on each fibre C_x of π_n, $\alpha|_{C_x} = \mathcal{O}(D)$ with $D = \sum d_i D_i$. By lemma 2 $H^1(C_x, \mathcal{O}(D)) = 0$, we have $R^1\pi_{n*}(\alpha) = 0$.

It remains to compute $R^0\pi_{n*}(\alpha)$. Since $H^1(C_x, \bigotimes_{i=1}^{n-1} L_i) = 0$, $H^0(C_x, \mathcal{O}(\sum_{i=1}^{n-1} d_i D_i))$ forms a vector bundle (call it H^0) on $\overline{M}_{0,n-1}$ by Grauert's theorem. Then $\pi_{n*}(\alpha)$ will be isomorphic to $H^0 \bigotimes (\bigotimes_{i=1}^{n-1} l_i^{\otimes d_i})$.

By definition, the fibre C_x is a tree of $\mathbf{P}^1$ with $n-1$ marked points $x_1, \cdots, x_{n-1}$. An element of $H^0(C_x, \mathcal{O}(D))$ is a meromorphic function with poles of order no more than d_i at x_i. It would be constant if there are no poles. Therefore an element of $H^0(C_x, \mathcal{O}(D))/(constants)$ is uniquely determined by the polar parts of the function at the marked points. By the rationality of each fibre the polar parts are independent and the space of polar parts at the marked point x_k is filtered by the degree of poles:

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_{d_k}$$

where $\mathcal{F}_i$ is the subspace of functions with poles of order no more than i. It is easy to see that the graded spaces $\mathcal{F}_{i+1}/\mathcal{F}_i$ are isomorphic to $T_{x_k}^{\otimes i}$. Therefore the vector bundle H^0 is *topologically* isomorphic to $(\underline{\mathbf{C}} \oplus l_1^{-1} \oplus l_1^{-2} \oplus \cdots \oplus l_1^{-d_1} \oplus \cdots \oplus l_{n-1}^{-1} \oplus \cdots \oplus l_{n-1}^{-d_{n-1}})$.

Combining all above, we have

$$\begin{aligned} &ch(R^0\pi_{n*}(\bigotimes_{i=1}^{n-1} L_i)) \\ =&(\prod_{i=1}^{n-1} e^{d_i c_1(l_i)})(1 + e^{-c_1(l_1)} + \cdots + e^{-d_1 c_1(l_1)} + \cdots + e^{-c_1(l_{n-1})} + \cdots + e^{-d_{n-1} c_1(l_{n-1})}). \end{aligned}$$

Now multiplying the above by $q_1^{d_1} \cdots q_{n-1}^{d_{n-1}}$ and summing over $(d_1, \cdots, d_{n-1}) \in \mathbf{Z}_+^n$ ($\mathbf{Z}_+ := \mathbf{N} \cup \{0\}$), we have

$$\begin{aligned} &\sum_{(d_1,\cdots,d_{n-1})} \prod_{i=1}^{n-1} e^{d_i c_1(l_i)} (1 + e^{-c_1(l_1)} + \cdots + e^{-d_{n-1}c_1(l_{n-1})}) q_1^{d_1} \cdots q_{n-1}^{d_{n-1}} \\ =& \sum_{\substack{(m_1,\cdots,m_{n-1}) \\ (d_1,\cdots,d_{n-1})}} e^{m_1 c_1(l_1)} \cdots e^{m_{n-1} c_1(l_{n-1})} q_1^{d_1} \cdots q_{n-1}^{d_{n-1}}, \end{aligned} \tag{12}$$

where either $m_i = d_i$ for all i or $m_i = d_i$ for all i except one for which $d_i - m_i \in \mathbf{N}$ is arbitrary. Thus (12) is equal to

$$\begin{aligned} &\sum_{(m_1,\cdots,m_{n-1}) \in \mathbf{Z}_+^n} (e^{c_1(l_1)} q_1)^{m_1} \cdots (e^{c_1(l_{n-1})} q_{n-1})^{m_{n-1}} [1 + \sum_{k=1}^{\infty} (q_1^k + \cdots + q_{n-1}^k)] \\ =&(1 + \frac{q_1}{1-q_1} + \cdots + \frac{q_{n-1}}{1-q_{n-1}})(\frac{1}{1 - q_1 e^{c_1(l_1)}}) \cdots (\frac{1}{1 - q_{n-1} e^{c_1(l_{n-1})}}), \end{aligned}$$

which is equal to the RHS of (11).

Q. E. D.

Since $\pi_n : \overline{M}_{0,n} \to M_{0,n-1}$ is a proper morphism of smooth varieties, we can apply the Grothendieck-Riemann-Roch theorem to any holomorphic line bundle α :

$$ch(\pi_{n*}(\alpha))Td(\overline{M}_{0,n-1}) = \pi_{n*}(ch(\alpha)Td(\overline{M}_{0,n})). \tag{13}$$

Proposition 1 can be restated as:

$$G(q_1, \cdots, q_n)|_{q_n=0} = (1 + \sum_{i=1}^{n-1} \frac{q_i}{1-q_i})G(q_1, \cdots, q_{n-1}). \tag{14}$$

Proposition 2. *On $\overline{M}_{0,n}$ the generating function (2) satisfies the following identity:*

$$\sum_{I\subset\{1,\cdots,n\}} (-1)^{|I|} \prod_{i\in I} \frac{1}{(1-q_i)} G(q)|_{\{q_i=0,\forall i\in I\}} = 0. \tag{15}$$

Proof. The above identity reads:

$$\int_{\overline{M}_{0,n}} Td(\overline{M}_{0,n})(\prod_{i=1}^{n}(\frac{1}{1-q_i e^{c_1(L_i)}} - \frac{1}{1-q_i})) = 0.$$

Since $dim_{\mathbf{C}}(\overline{M}_{0,n}) = n-3 < n$, the identity follows from the fact that $(1-qe^c)^{-1}-(1-q_i)^{-1}$ is divisible by c and dimension counting.

Q. E. D.

Now $G(q_1, \cdots, q_n)$ with (at least) one $q_i = 0$ was calculated in Proposition 1 as (constant)$G(q_1, \cdots, q_{n-1})$, which is known by induction hypothesis. Combining (14) and (15) we have

$$\begin{aligned} & G(q_1, \cdots, q_n) \\ =& \sum_{j=1}^{n} (1 + \sum_{i\neq j} \frac{q_i}{1-q_i})^{n-3} \prod_{i=1}^{n} \frac{1}{1-q_i} - \sum_{j<k} (1 + \sum_{j\neq i\neq k} \frac{q_i}{1-q_i})^{n-3} \prod_{i=1}^{n} \frac{1}{1-q_i} \\ & + \cdots\cdots + (-1)^{n-1} \prod_{i=1}^{n} \frac{1}{1-q_i}. \end{aligned}$$

Thus we are left to prove the combinatorial identity ($a_i := \frac{q_i}{1-q_i}$) :

$$(1 + \sum_{i=1}^{n} a_i)^{n-3} = \sum_{j_1=1}^{n} (1 + \sum_{i\neq j_1} a_i)^{n-3} - \sum_{j_1<j_2} (1 + \sum_{j_1\neq i\neq j_2} a_i)^{n-3} + \cdots + (-1)^{n-1}1,$$

which is easily verified by elementary algebra. For example it is the finite difference version of the identity:

$$\frac{\partial^n}{\partial x_1 \cdots \partial x_n}(1 + x_1 + x_2 + \cdots + x_n)^{n-3} = 0$$

at $(0, 0, \cdots, 0)$.

Remarks :

(i) The formulas of Euler characteristics (coefficient of $G(q_1, \cdots, q_n)$) are rather complicated. It is remarkable that they can be packed in a very elegant generating function.

(ii) As we have mentioned in the introduction, there are some evidences which support our guess on the vanishing of $H^i(\overline{M}_{0,n}, \bigotimes L_i^{d_i})$ for $i \geq 1$. First, in the case of $\overline{M}_{0,4}$ and $\overline{M}_{0,5}$ it can be explicitly computed. Second, if the line bundle $\bigotimes_{i=1}^n L_i^{\otimes d_i}$ consists of only $n-1$ bundles, i.e. some $d_i = 0$, then the vanishing of higher cohomology groups can be proved by the same arguments in the proof of proposition 2 and simple spectral sequence argument. Even in the case when some $d_i = 1$ it can be proved by the same method (some modifications are necessary) plus the residue theorem. But when $\min\{d_i\} \geq 2$ the computation becomes very complicated and we don't know how to proceed.

(iii) The above proof is based on a suggestion by A. Givental. Our original proof was done by a "term by term" argument, i. e. computing the correlation functions instead of the generating function itself. In fact we can derive a reduciton formula relating the correlation functions (1) on $\overline{M}_{0,n}$ in terms of those on $\overline{M}_{0,n-1}$, similar to the derivation [DW] of the correlation functions (4) in the theory of two-dimensional gravity. Not surprisingly we have to deal with very complicated combinatorial identities and cancellation process. One of the improvement of the present proof is that we derive the generating functions directly, and therefore drastically reduce the complexity of combinatorial identities.

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