EULER CHARACTERISTICS OF UNIVERSEAL COTANGENT LINE BUNDLES ON $\overline{M}_{1,n}$

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ABSTRACT. An effective algorithm of computing the Euler characteristics $\chi(\overline{M}_{1,n}, \otimes L_i^d)$ is given, where $\overline{M}_{1,n}$ is the moduli stack of $n$-pointed stable curves of genus one and $L_i$ are the universal cotangent line bundles. This work is a sequel to [8].

In addition, we give a simple proof of Pandharipande’s vanishing theorem $H^j(\overline{M}_{0,n}, \otimes L_i^d) = 0$ for $j \geq 1, d_i \geq 0$.

0. INTRODUCTION

Let $\overline{M}_{1,n}$ be the moduli stack of $n$-pointed genus 1 stable curves, $\mathcal{O}$ its structure sheaf, $\mathcal{H}$ the Hodge bundle, and $L_i, 1 \leq i \leq n$, the universal cotangent line bundles at the $i$-th marked point. The main result of this paper is the following theorem.

Theorem 0.1. There is an effective algorithm of computing the Euler characteristics

$$\chi_{d,d_1,...,d_n} := \chi(\overline{M}_{1,n}, \mathcal{H}^{-d} \otimes \bigotimes_{i=1}^n L_i^{d_i}), \quad d, d_i \geq 0.$$  

The details of this algorithm is presented in Section 2.

This work is a sequel to [8], where we calculate the Euler characteristics $\chi(\overline{M}_{0,n}, \otimes L_i^d)$ at genus zero. These are our preliminary attempts in search of a $K$-theoretic version of Witten–Kontsevich’s theory of two-dimensional topological gravity. In the Witten-Kontsevich theory, the correlators are the intersection numbers of tautological classes on the moduli spaces of stable curves

$$\int_{\overline{M}_{g,n}} \psi_1^{d_1} \cdots \psi_R^{d_n}. \quad (0.1)$$

The natural $K$-theoretic version of intersection numbers (i.e. pushforward to a point in cohomology theory) are the Euler characteristics (i.e. pushforward to a point in $K$-theory). Since the Witten–Kontsevich theory states that a generating function of (0.1) is the $\tau$-function of the $KdV$ hierarchy,

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it is reasonable to surmise that a similar generating function in $K$-theory could be a $\tau$-function of a version of discrete $KdV$ hierarchy. Note that the phenomenon of replacing differential equations in quantum cohomology by finite difference equations in quantum $K$-theory were observed in earlier examples [3] and only very recently demonstrated to hold in general [4].

Furthermore, since Witten–Kontsevich theory is the Gromov–Witten theory for the target space being a point, it is natural to consider these calculations as basic ingredients for the quantum $K$-theory developed in [2] and [9]. In the calculation of quantum $K$-invariants at genus one via localization, the Hodge bundle will natural appear. It is therefore reasonable to consider slightly more general correlators, which have the additional benefits of facilitating the induction process of our algorithm.

Our strategy of proving Theorem 0.1 is, very roughly, to apply the orbifold Riemann–Roch theorem to

$$\chi^\prime := \chi \left( \mathcal{M}_{1,n}, \mathcal{H}^{-d} \bigotimes_{i=1}^{n} (L_i^{d_i} - \mathcal{O}) \right), \quad d, d_i \geq 0.$$ 

By carefully examining and performing computations on the twisted sectors, we were able to determine the $\chi^\prime$. In doing so, we find the use of generating functions essential. This can be found in Section 2, starting with Equation (2.1). It is then not difficult to see that one can determine $\chi$ on $\mathcal{M}_{1,n}$ by $\chi^\prime$ and $\chi$ on $\mathcal{M}_{1,n-1}$. Hence, we can reduce all calculations to $n = 1$ case, whose generating function we compute explicitly in Lemma 2.8 and Proposition 2.9. Note that the generating function in $n = 1$ is a rational function, as in the case of genus zero case. We expect the generating functions of $\chi_{d,d_1,...,d_n}$ to be rational functions as well, but are not able to find the correct form. We did, however, perform a consistency check: Our algorithm produces $\chi_{d,d_1,...,d_n}$ as integers, even though the intermediate steps requires rational numbers, which are the consequences of the orbifold Riemann–Roch and the stack structures of $\mathcal{M}_{1,n}$.

Indeed, the $n = 1$ case is closely related to the theory of modular forms. Theorem 0.1 can be considered as a vast generalization of the following well-known fact.

**Proposition 0.2** (See Lemma 2.8 and Proposition 2.9).

$$\chi \left( \mathcal{M}_{1,1}, \frac{1}{1 - qL_1} \right) = \frac{1}{(1 - q^4)(1 - q^6)}.$$ 

Since $\mathcal{M}_{1,1}$ is the moduli stack of elliptic curves, and sections of $L_1^k$ are the modular forms of weight $2k$, Proposition 0.2 can be considered as a rephrase of the classical result that the space of the modular forms are generated by a weight four and a weight six modular forms.
Another result included in this paper, in Appendix A, is a new proof of Pandharipande’s vanishing theorem [11] at genus zero.

**Theorem 0.3** ([11]).

\[ H^j(\overline{\mathcal{M}}_{0,n}, \otimes_{i=1}^n L^{d_i}) = 0 \]

for \( j \geq 1 \) and \( d_i \geq 0 \).

Our proof is comparably simpler and shorter, and do not use M. Kapranov’s results on \( \overline{\mathcal{M}}_{0,n} \) [5]. Only basic definitions and elementary manipulation of spectral sequences are used.

This paper is organized as follows. In Section 1 we recall the necessary background on the structure of the inertia stack of \( \overline{\mathcal{M}}_{1,n} \), mostly quoting [10]. We then formulate a more precise version of the reduction algorithm in Section 2, and prove Theorem 0.1 there. In Appendix A the (new) proof of Theorem 0.3 is given.

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1. **Preliminaries**

We work over the ground field \( \mathbb{C} \).

1.1. **Twisted sectors of \( \overline{\mathcal{M}}_{1,n} \).** We summarize the results we need concerning the inertia stack of \( \overline{\mathcal{M}}_{1,n} \) in [10].

For a DM stack \( \mathcal{X} \), recall a Spec \( \mathbb{C} \) point of its inertia stack \( I\mathcal{X} \) is given by a pair \((x, g)\) with \( x \in \mathcal{X}(\text{Spec} \mathbb{C}) \) and \( g \in \text{Aut}_\mathcal{X}(x) \). \( \mathcal{X} \) is naturally viewed as a component of \( I\mathcal{X} \) consists of point \((x, g)\) with \( g \) trivial, we denote this component by \((\mathcal{X}, \text{Id})\). A twisted sector is a connected component of \( I\mathcal{X} \) disjoint from \((\mathcal{X}, \text{Id})\). By a component, we mean an open and closed sub-stack.

**Theorem 1.1** ([10] Theorem 3.22, 3.24). The twisted sectors of \( I\overline{\mathcal{M}}_{1,n} \) come from either of the following two sources:

1. the closure of a twisted sector of \( IM_{1,n} \) in \( IM_{1,n} \), or
2. A twisted sector of \( I(\overline{\mathcal{M}}_{0,KU} \times \overline{\mathcal{M}}_{1,KcU}) \) via \( I\Delta_K \).

Here \( \Delta_K : \overline{\mathcal{M}}_{0,KU} \times \overline{\mathcal{M}}_{1,KcU} \to \overline{\mathcal{M}}_{1,n} \) is the closed immersion gluing the marked points \( * \), \( K \) is a subset of \( [n] \) with \( |K| \geq 2 \), and \( K^c \) its complement. \( I\Delta_K : I(\overline{\mathcal{M}}_{0,KU} \times \overline{\mathcal{M}}_{1,KcU}) \to I\overline{\mathcal{M}}_{1,n} \) is the induced closed immersion between the corresponding inertia stacks.

As \( I(\overline{\mathcal{M}}_{0,KU} \times \overline{\mathcal{M}}_{1,KcU}) \simeq \overline{\mathcal{M}}_{0,KU} \times IM_{1,KcU} \), type (2) twisted sectors are built up from type (1) twisted sectors.

The analysis of type (1) twisted sectors in [10] starts with a known description of \( \overline{\mathcal{M}}_{1,1} \).
Proposition 1.2. \( \overline{\mathcal{M}}_{1,1} \) is equivalent to the weighted projective space \( \mathbb{P}(4,6) \).

We briefly recall the equivalence appeared in [10] Theorem 3.9, as we will need it to do explicit calculations on \( \overline{\mathcal{M}}_{1,1} \), it also serves to motivate the notations used in [10](Notation 3.11, 3.14; Definition 3.15, 3.18) that we follow.

Let \( U = \mathbb{A}^2 - (0,0) \) with \( \mathbb{C}^* \) action: \( \lambda \cdot (a, b) = (\lambda^4 a, \lambda^6 b) \), where \( \lambda \in \mathbb{C}^* \), \( (a,b) \in U \). The equivalence from \( \mathbb{P}(4,6) := [U/\mathbb{C}^*] \) to \( \overline{\mathcal{M}}_{1,1} \) is induced from a \( \mathbb{C}^* \)-equivariant family of 1 pointed genus one stable curves \( C \to U \).

\[
C = \{(a,b) \times [x:y:z] \in U \times \mathbb{P}^2 \mid y^2 z = x^3 + axz^2 + bz^3\},
\]

with section
\[
s : U \to U \times [0,1,0] \subset C,
\]
the \( \mathbb{C}^* \) action is given by
\[
\lambda \cdot ((a,b) \times [x:y:z]) = (\lambda^4 a, \lambda^6 b) \times [\lambda^2 x : \lambda^3 y : z].
\]

Denote by

- \( A_k \): the component of \( \overline{\mathcal{M}}_{1,k} \) consisting of pairs \( (x,g) \) with \( g \) of order 2, here \( 1 \leq k \leq 4 \).
- \( C_4 \): the 1-pointed curve \( \{[x:y:z] \in \mathbb{P}^2 \mid y^2 z = x^3 + xz^2\} \) with \( [0:1:0] \) marked. Its automorphism group is generated by \( i = \sqrt{-1} \).
- \( C_6 \): 1-pointed curve \( \{[x:y:z] \in \mathbb{P}^2 \mid y^2 z = x^3 + z^3\} \) with \( [0:1:0] \) marked. Its automorphism group is generated by \( e = \exp(\pi i / 6) \).
- \( C_4', C_6' \): \( C_4 \) with a 2nd marked point \( [0:0:1] \).
- \( C_6', C_6'' \): \( C_6 \) with a 3rd marked point \( [0:-1:1] \).

Theorem 1.3 ([10] Corollary 3.16).

1. \( I \mathcal{M}_{1,5} = (\mathcal{M}_{1,5}, \text{Id}) \), \( n \geq 5 \).
2. The twisted sectors of \( I \mathcal{M}_{1,k}, k \leq 4 \) are of dimension 1 or 0. \( A_k \) is the only 1 dimensional twisted sector. All the 0 dimensional twisted sectors are of the form \( B\mu_r \), and they are determined by
   - \( (C_4, i), (C_4, -i), (C_6, e), (C_6, e^2), (C_6, e^3) \) for \( \mathcal{M}_{1,1} \).
   - \( (C_4', i), (C_4', -i), (C_6', e^2), (C_6', e^4) \) for \( \mathcal{M}_{1,2} \).
   - \( (C_6'', e^2), (C_6'', e^4) \) for \( \mathcal{M}_{1,3} \).

Remark 1.4. Given \( x \in \mathcal{X}(\text{Spec} \mathbb{C}) \) and an order \( r \) element \( g \in \text{Aut}_X(x) \), the pair \( (x,g) \) determines a representable morphism from \( B\mu_r \) to \( \mathcal{X} \). (see [1] 3.2) As \( B\mu_r \) is proper, it is closed in \( I \mathcal{M}_{1,k} \).

Theorem 1.5 ([10] Corollary 3.9, 3.19).

Let \( \overline{A}_k \) be the closure of \( A_k \) in \( I \mathcal{M}_{1,k} \), then

1. \( \overline{A}_1 \) is isomorphic to \( \overline{\mathcal{M}}_{1,1} \).
2. \( \overline{A}_2 \subset I \mathcal{M}_{1,2} \) is isomorphic to \( \mathbb{P}(2,4) \).
3. \( \overline{A}_3 \subset I \mathcal{M}_{1,3} \) is isomorphic to \( \mathbb{P}(2,2) \).
• $\overline{\mathcal{M}}_{4} \subset I\overline{\mathcal{M}}_{1,4}$ is isomorphic to $\mathbb{P}(2,2)$.

(2) When viewed as closed substack of $\overline{\mathcal{M}}_{1,k}$, $\overline{\mathcal{A}}_{k}$ does not intersect with the boundary divisors $\Delta_{K}$ for any $K \subset [k], 2 \leq k \leq 4$.

1.2. Riemann-Roch formula for Stacks. We recall the Riemann-Roch formula in a version needed for this paper, adopted from Appendix A of [13].

**Theorem 1.6** ([6],[12] Corollary 4.13). Let $\mathcal{X}$ be a smooth, proper Deligne-Mumford stack with quasi-projective coarse moduli space, $E$ a vector bundle on $\mathcal{X}$. Assume $\mathcal{X}$ has the resolution property, i.e. every coherent sheaf is a quotient of a vector bundle, then we have the following formula for the Euler characteristics of $E$:

$$\chi(\mathcal{X}, E) = \int_{I\mathcal{X}} \tilde{C}h(E) \tilde{T}d(\mathcal{X}).$$

Here

- $I\mathcal{X}$ is the inertial stack of $\mathcal{X}$, with projection $p : I\mathcal{X} \to \mathcal{X}$.
- $\tilde{C}h(E) \in H^{*}(I\mathcal{X})$ is the Chern character of the bundle $\rho(p^{*}E)$.
- $\rho(F) := \sum_{\xi} \xi F(\xi) \in K^{0}(I\mathcal{X}), F(\xi)$ is the eigenbundle of $F$ with eigenvalue $\xi$.
- $\tilde{T}d(\mathcal{X}) = \frac{Td(I\mathcal{X})}{\tilde{C}h(p^{*}\lambda_{-1}(N_{I\mathcal{X}/\mathcal{X}}))}$, where $Td$ and $Ch$ are the usual Todd class and Chern character. $N_{I\mathcal{X}/\mathcal{X}}$ is the normal bundle for $p$, and $N^{\vee}$ is the dual of $N$.
- $\lambda_{-1}(V) := \sum_{a \geq 0} (-1)^{a} \Lambda^{a}V$ is the $\lambda_{-1}$ operation in K-theory. If $V = \bigoplus V_{i}$ is direct sum of line bundles $V_{i}$, then $\lambda_{-1}(V) = \prod_{i}(1 - V_{i})$.

**Remark 1.7.** $\overline{\mathcal{M}}_{1,n}$ satisfies the resolution property. See, e.g. [7] Proposition 5.1.

1.3. String Equation.

**Proposition 1.8** ([9] Sec.4.4). Let $\pi : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n-1}$ be the forgetful map forgetting the n-th marked point, then in terms of generating functions with variables $q, q_{i}, 1 \leq i < n$, we have, for $g = 0$,

$$\pi_{*}(\prod_{i < n} \frac{1}{1 - q_{i}L_{i}}) = \prod_{i < n} \frac{1}{1 - q_{i}L_{i}} \cdot (1 + \sum_{i < n} \frac{q_{i}}{1 - q_{i}}),$$

and when $g > 0$,

$$\pi_{*}(\frac{1}{1 - qH^{-1}} \prod_{i < n} \frac{1}{1 - q_{i}L_{i}}) = \frac{1}{1 - qH^{-1}} \prod_{i < n} \frac{1}{1 - q_{i}L_{i}} \cdot (1 - H^{-1} + \sum_{i < n} \frac{q_{i}}{1 - q_{i}}).$$

Here $\pi_{*}$ is the K-theoretic pushforward, $d, d_{i}$ are nonnegative integers.
2. Euler Characteristics of Universal Cotangent Line Bundles

In this section, we give an algorithm to compute

\[ \chi \left( \overline{\mathcal{M}}_{1,n}, \mathcal{H}^{-d} \bigotimes_{i=1}^{n} L_{i}^{d_{i}} \right), d_{i} \geq 0. \]

It is more efficient to encode these numbers into a generating function

\[ X_{n} := \chi \left( \overline{\mathcal{M}}_{1,n}, \frac{1}{1 - q_{H}^{-1} \prod_{i=1}^{n} \left( \frac{1}{1 - q_{i} L_{i}} \right)} \right) \]

(2.1)

We will first show that the calculation of \( X_{n} \) can be reduced to that of \( X_{n-1} \) if another generating function \( \Phi_{n} \) in (2.2) can be calculated. We then explicitly determine \( \Phi_{n} \) and \( X_{1} \).

2.1. Reduction from \( \overline{\mathcal{M}}_{1,n} \) to \( \overline{\mathcal{M}}_{1,n-1} \). Let \( \Phi_{n} \) be the generating function

\[ \Phi_{n} := \chi \left( \overline{\mathcal{M}}_{1,n}, \frac{1}{1 - q_{H}^{-1} \prod_{i=1}^{n} \left( \frac{1}{1 - q_{i} L_{i}} \right)} \right) \]

(2.2)

Lemma 2.1. When \( n > 1 \), \( X_{n} \) is determined by \( \Phi_{n} \) and \( X_{n-1} \). More precisely, we have

\[ X_{n}(q, q_{1}, \ldots, q_{n}) = \Phi_{n}(q, q_{1}, \ldots, q_{n}) + \sum_{I \subseteq [n], I \neq \emptyset} (-1)^{|I|+1} \prod_{i \in I} \frac{1}{1 - q_{i}} \cdot \left( X_{n-1}(q_{j}, \{ j, j \notin I \}, 0, \ldots, 0) \right) \left( \frac{1}{q} + \prod_{j \notin I} \frac{q_{j}}{1 - q_{j}} \right) \]

\[ + \frac{1}{q} X_{n-1}(0, \{ q_{j}, j \notin I \}, 0, \ldots, 0) \right) \]

For the last line, note that \( X_{n-1}(q, q_{1}, \ldots, q_{n-1}) \) is a symmetric function of the variables \( q_{1}, q_{2}, \ldots, q_{n-1} \), and it is evaluated at \( \{ q_{j}, j \notin I \} \) with \( |I| - 1 \) zeros.

Proof. This follows directly from the definition and the string equation. Expand the product \( \prod_{i=1}^{n} \left( \frac{1}{1 - q_{i, t}} - \frac{1}{1 - q_{i, u}} \right) \) in \( \Phi_{n} \) we have

\[ \Phi_{n} = X_{n} + \sum_{I \subseteq [n], I \neq \emptyset} (-1)^{|I|} \prod_{i \in I} \frac{1}{1 - q_{i}} \cdot \chi \left( \overline{\mathcal{M}}_{1,n}, \frac{1}{1 - q_{H}^{-1} \prod_{j \notin I} \left( \frac{1}{1 - q_{j} L_{j}} \right)} \right). \]
By the string equation
\[
\chi\left(\overline{\mathcal{M}}_{1,n'} \begin{array}{c}
1 \\
1 - q\mathcal{H}^{-1} \\
\prod_{j \notin I} 1 - q_j
\end{array} \right)
\]
\[
= \chi\left(\overline{\mathcal{M}}_{1,n-1} \begin{array}{c}
1 \\
1 - q\mathcal{H}^{-1} \\
\prod_{j \notin I} 1 - q_j
\end{array} \cdot (1 - \mathcal{H}^{-1} + \sum_{j \notin I} \frac{q_j}{1 - q_j})\right),
\]
which is
\[
X_{n-1}(q, \{q_j, j \notin I\}, 0, \ldots, 0)(1 - \frac{1}{q} + \sum_{j \notin I} \frac{q_j}{1 - q_j})
\]
\[
+ \frac{1}{q} X_{n-1}(0, \{q_j, j \notin I\}, 0, \ldots, 0).
\]
From here it is easy to see the lemma holds.

2.2. Calculation of \(\Phi_n\). By the Riemann-Roch formula (Theorem 1.6), we have
\[
\Phi_n = \int_{\overline{\mathcal{M}}_{1,n}} \overline{\text{Ch}}\left(\begin{array}{c}
1 \\
1 - q\mathcal{H}^{-1} \\
\prod_{i=1}^n \left(1 - \frac{1}{q_i} - \frac{1}{1 - q_i}\right)\end{array}\right) \overline{Td}(\overline{\mathcal{M}}_{1,n}).
\]

As the inertia stack \(\overline{\mathcal{M}}_{1,n}\) is the disjoint union of the distinguished component \((\overline{\mathcal{M}}_{1,n}, \text{Id})\) and its twisted sectors, the integral is the sum of the contributions from these components.

**Proposition 2.2.** The contribution to \(\Phi_n\) from \((\overline{\mathcal{M}}_{1,n}, \text{Id})\) is
\[
\frac{(n-1)!}{24(1-q)} \prod_{i=1}^n \left(\frac{q_i}{1-q_i}\right)^2.
\]

**Proof.** On \((\overline{\mathcal{M}}_{1,n}, \text{Id})\), \(\overline{\text{Ch}}, \overline{Td}\) reduces to the usual \(\text{Ch}, \text{Td}\),
\[
\text{Ch}\left(\begin{array}{c}
1 \\
1 - q\mathcal{H}^{-1} \\
\prod_{i=1}^n \left(1 - \frac{1}{q_i} - \frac{1}{1 - q_i}\right)\end{array}\right)
\]
\[
= \frac{1}{1 - q} \prod_{i=1}^n \frac{q_i}{(1-q_i)^2} \prod_{i=1}^n c_1(L_i) + \text{higher degree terms}.
\]

Applying the dilaton equation
\[
\int_{\overline{\mathcal{M}}_{1,n}} c_1(L_1) \cdots c_1(L_n) = (n-1) \int_{\overline{\mathcal{M}}_{1,n-1}} c_1(L_1) \cdots c_1(L_{n-1}),
\]
and
\[
\int_{\overline{\mathcal{M}}_{1,1}} c_1(L_1) = \frac{1}{24},
\]
we can now evaluate the integral

\[
\int_{\overline{M}_{1,n}} Ch \left( \frac{1}{1 - q} \prod_{i=1}^{n} \left( \frac{1}{1 - q_i L_i} - \frac{1}{1 - q_i O} \right) \right) Td(\overline{M}_{1,n})
\]

\[
= \frac{(n - 1)!}{24(1 - q)} \prod_{i=1}^{n} \frac{q_i}{(1 - q_i)^2}.
\]

The proposition is proved. \(\square\)

**Proposition 2.3.** The contribution to \(\Phi_n\) from the twisted sectors of type (2) in Theorem 1.1, i.e. from \(I_\Delta K\), is zero.

**Proof.** By our construction, such a twisted sector is the product of \(\overline{M}_{0,KU,\bullet}\) and a twisted sector \(T\) of \(\overline{M}_{1,KU,\bullet}\).

The natural map \(\overline{M}_{0,KU,\bullet} \times T \to \overline{M}_{1,n}\) factors through

\(\Delta_K : \overline{M}_{0,KU,\bullet} \times \overline{M}_{1,KU,\bullet} \to \overline{M}_{1,n}\).

We quote some known results:

- The dual of the normal bundle for \(\Delta_K\) is \(pr^*_1(L_\bullet) \otimes pr^*_2(L_\bullet)\). Here \(pr_i\) is the projection of \(\overline{M}_{0,KU,\bullet} \times \overline{M}_{1,KU,\bullet}\) onto its \(i\)-th factor.
- \(\Delta^*_K(L_i)\) is \(pr^*_1(L_i)\) for \(i \in K\), and is \(pr^*_2(L_i)\) for \(i \notin K\).
- \(\Delta^*_K(\mathcal{H}) = pr^*_2(\mathcal{H})\).

Using these results, it is then straightforward to see that pushing forward the integrand

\[
\overline{Ch}(\frac{1}{1 - q} \prod_{i=1}^{n} \left( \frac{1}{1 - q_i L_i} - \frac{1}{1 - q_i O} \right))(\overline{M}_{1,n})
\]

gives us a class which has a factor \(\prod_{i \in K} c_1(L_i)\) coming from \(\overline{Ch}(\prod_{i \in K}(\frac{1}{1 - q_i L_i} - \frac{1}{1 - q_i O}))\). As the degree of \(\prod_{i \in K} c_1(L_i)\) already exceeds the dimension of \(\overline{M}_{0,KU,\bullet}\), the contribution is zero. \(\square\)

**Proposition 2.4.** For \(2 \leq k \leq 4\), the contribution from \(A_k\) is

\[
(-1)^k \frac{1}{24(1 + q)} \prod_{i=1}^{k} \frac{q_i}{1 - q_i^2} \cdot (11 + \frac{2q}{1 + q} - \sum_{i=1}^{n} \frac{2q_i}{1 + q_i}) \cdot d_k,
\]

where \(d_k\) is 6, 6, 3 for \(k = 4, 3, 2\), respectively. The numbers \(d_k\) are the degree of the maps \(\overline{A}_k \to \overline{M}_{1,1}\) forgetting all but one marked point.
Proof. On $\overline{A}_k$, we have
\[
\tilde{Ch} \left( \frac{1}{1-q} \prod_{i=1}^{k} \left( \frac{1}{1-q_i L_i} - \frac{1}{1-q_i O} \right) \right)
= \frac{1}{1+qe^{c_1(L)}} \prod_{i=1}^{k} \left( \frac{1}{1+qe^{c_1(L)}} - \frac{1}{1-q_i} \right)
= (-2)^k \frac{1}{1+q} \prod_{i=1}^{k} \frac{q_i}{1-q_i} \left( 1 + \frac{q}{1+q} c_1(H) + \sum_{i=1}^{k} \frac{1-q_i}{2(1+q)} c_1(L_i) \right),
\]
and
\[
Ch \left( \rho \circ (\Lambda_{-1}(N_{\overline{A}_k/M_{1,k})) \right) = 2^{k-1} \left( 1 + \frac{1}{2} c_1(N_{\overline{A}_k/M_{1,k}}) \right).
\]
Note that over $\overline{A}_k$ the eigenvalues involved in $\tilde{Ch}$ must be $-1$, as the nontrivial automorphism is of order 2, also there is no higher degree terms as $\overline{A}_k$ is 1 dimensional.

Thus
\[
\int_{\overline{A}_k} \tilde{Ch} \left( \frac{1}{1-q} \prod_{i=1}^{k} \left( \frac{1}{1-q_i L_i} - \frac{1}{1-q_i O} \right) \right) \tilde{Td}(M_{1,k})
= (-1)^k \frac{1}{1+q} \prod_{i=1}^{k} \frac{q_i}{1-q_i} \cdot \left( \frac{2q}{1+q} \int_{\overline{A}_k} c_1(H) + \sum_{i=1}^{k} \frac{1-q_i}{1+q} \int_{\overline{A}_k} c_1(L_i) + \int_{\overline{A}_k} c_1(TM_{1,k}) \right).
\]

It is easy to see
\[
\int_{\overline{A}_k} c_1(H) = d_k \int_{M_{1,1}} c_1(H) = \frac{d_k}{24},
\]
by considering a map $\overline{A}_k \subset \overline{M}_{1,k} \to \overline{M}_{1,1}$ forgetting all but one marked point,
\[
\int_{\overline{A}_k} c_1(L_i), 1 \leq i \leq k, \text{ and } \int_{\overline{A}_k} c_1(TM_{1,k}).
\]

are determined by Corollary 2.6.

\[\square\]

**Lemma 2.5.** Let $\pi : \overline{M}_{1,n+1} \to \overline{M}_{1,n}$ be the forgetful map forgetting the $(n+1)$-th marked point, then
\[
c_1(L_j) = \pi^* c_1(L_j) + \Delta_{\{j,n+1\}}, 1 \leq j \leq n;
\]
\[
c_1(TM_{1,n+1}) = \pi^* c_1(TM_{1,n}) - c_1(L_{n+1}) + \sum_{1 \leq i \leq n} \Delta_{\{j,n+1\}}.
\]
Proof. Recall for \( \pi : \overline{\mathcal{M}}_{1,n+1} \rightarrow \overline{\mathcal{M}}_{1,n} \) we have
\[
L_j = \pi^* L_j (\Delta_{j(n+1)}), 1 \leq j \leq n; \quad L_{n+1} = \omega_\pi (\sum_{1 \leq j \leq n} \Delta_{j(n+1)}),
\]
where \( \omega_\pi = \omega_{\overline{\mathcal{M}}_{1,n+1}} \otimes \pi^* \omega_{\overline{\mathcal{M}}_{1,n}}^{-1} \) is the relative dualizing sheaf for \( \pi \). Taking the first Chern class of these equations proves the lemma. \( \Box \)

Corollary 2.6. \[
\int_{A_k} c_1(L_j) = \frac{d_k}{24}, 1 \leq j \leq k. \quad \int_{A_k} c_1(T \overline{\mathcal{M}}_{1,k}) = \frac{(11 - k)d_k}{24}.
\]
Proof. This can be easily proved by applying the above lemma and Theorem 1.5 (2) to a forgetful map \( A_k \rightarrow \overline{\mathcal{M}}_{1,1} \). \( \Box \)

Proposition 2.7. The contribution to \( \Phi_n \) from the integral over zero dimensional twisted sectors are the following.

- the contribution from \( (C_{4'}, i) \sqcup (C_{4'}, -i) \) is
\[
\frac{1}{4} \prod_{j=1,2} \frac{q_j}{(1 - q_j)} \cdot \frac{1 - q + q_1 + q_2 - q_1 q_2 + q q_1 + q_1 q_2 + q q_1 q_2}{(1 + q^2)(1 + q_1^2)(1 + q_2^2)}.
\]

- the contribution from \( (C_{6'}, e^2) \sqcup (C_{6'}, e^4) \) is
\[
\frac{1}{3} \prod_{j=1,2} \frac{q_j}{(1 - q_j)} \cdot \frac{1 - q + (q + 2)(q_1 + q_2) + (2q + 1)q_1 q_2}{(1 + q + q^2)(1 + q_1 + q_1^2)(1 + q_2 + q_2^2)}.
\]

- the contribution from \( (C_{6'}, e^2) \sqcup (C_{6'}, e^4) \) is
\[
-\frac{1}{3} \prod_{j=1,2,3} \frac{q_j}{(1 - q_j)} \cdot \frac{1 - q + (q + 2)(q_1 + q_2 + q_3) + (2q + 1)(q_1 q_2 + q_1 q_3 + q_2 q_3) + (q - 1)q_1 q_2 q_3}{(1 + q + q^2)(1 + q_1 + q_1^2)(1 + q_2 + q_2^2)(1 + q_3 + q_3^2)}.
\]

Proof. To simplify the notation, we will use \((C, \lambda)\) to denote a twisted sector.

We need to determine the eigenvalues of the bundles involved in the integrand. The fiber of \( L_j \) at a smooth \( n \) pointed curve \( \{ C, x_1, x_2, \ldots, x_n \} \) is \( T^*_{x_j} C \), so the eigenvalue is determined by the action of the automorphism on this cotangent space. It is \( \lambda \) on \((C, \lambda)\). The eigenvalue for \( \mathcal{H} \) is also \( \lambda \), as it is the pullback of \( L_1 \) on \( \overline{\mathcal{M}}_{1,1} \) via a forgetful map. For \((C_{4'}, \lambda), (C_{6'}, \lambda)\) of \( T^* \overline{\mathcal{M}}_{1,1} \), explicit calculation tells us the eigenvalue of \( T^* \overline{\mathcal{M}}_{1,1} \) is \( \lambda^2 \). It is then not hard to show, using a forgetful map to \( \overline{\mathcal{M}}_{1,1} \), that the eigenvalues of \( T^* \overline{\mathcal{M}}_{1,2} \) are \( \lambda, \lambda^2 \) for \((C_{4'}, \lambda)\) and \((C_{6'}, \lambda)\), the eigenvalues of \( T^* \overline{\mathcal{M}}_{1,3} \) are \( \lambda, \lambda, \lambda^2 \) for \((C_{6'}, \lambda)\).

From the analysis above, on the twisted sector \((C, \lambda)\) of \( \overline{\mathcal{M}}_{1,n} \) we have
Calculation for Proposition 2.9.

By Serre duality, 

\[ \widetilde{\chi} = \frac{1}{1 - qH^{-1}} \prod_{i=1}^{n} \left( \frac{1}{1 - q_{i} L_{i}} - \frac{1}{1 - q_{i} O} \right) = \frac{(\lambda - 1)^{n}}{1 - q\lambda - 1} \prod_{i=1}^{n} \frac{q_{i}}{(1 - q_{i})(1 - q_{i}\lambda)} \]

and

\[ \widetilde{T}d(\mathcal{M}_{1,n}) = \frac{1}{(1 - \lambda^{2})(1 - \lambda)^{n-1}} = \frac{1}{(1 + \lambda)(1 - \lambda)^{n}}. \]

The sum of the integral on \((C_{4}, i) \sqcup (C_{4}, -i)\) is then

\[ \frac{1}{4} \sum_{\lambda = i, -i} \frac{1}{(1 - q\lambda^{-1})(1 + \lambda)} \prod_{i=1,2} \frac{q_{i}}{(1 - q_{i})(1 - q_{i}\lambda)}, \]

which equals

\[ \frac{1}{4} \prod_{j=1,2} \frac{q_{j}}{(1 - q_{j})} \cdot \frac{1 - q + q_{1} + q_{2} - q_{1}q_{2} + q_{1}q_{2} + q_{2}q_{1}q_{2}}{(1 + q^{2})(1 + q^{2})(1 + q^{2})}. \]

The remaining cases also follow directly from our formula of \(\widetilde{\chi}, \widetilde{T}d\). \(\square\)

2.3. Calculation for \(X_{1}\). Under the isomorphism \(\mathcal{M}_{1,1} \simeq \mathbb{P}(4, 6), \mathcal{H}, L_{1}\) all corresponds to \(\mathcal{O}(1)\), so

\[ \chi(\mathcal{M}_{1,1}, \mathcal{H}^{-d} \otimes L_{1}^{d}) = \chi(\mathbb{P}(4, 6), \mathcal{O}(d - d)), \]

and we see \(X_{1}\) is determined by \(\chi(\mathbb{P}(4, 6), \mathcal{O}(k)), k \in \mathbb{Z}\).

**Lemma 2.8.** Let \(h^{0}(\mathcal{O}(k)) = \dim_{\mathbb{C}} H^{0}(\mathbb{P}(4, 6), \mathcal{O}(k))\), then

\[ \sum_{k=0}^{\infty} h^{0}(\mathcal{O}(k))q^{k} = \frac{1}{(1 - q^{4})(1 - q^{6})}, \]

and \(h^{0}(\mathcal{O}(k)) = 0\) if \(k < 0\).

**Proof.** The section of \(\mathcal{O}(k)\) on \(\mathbb{P}(4, 6)\) corresponds to polynomials \(f(x, y) \in \mathbb{C}[x, y]\) such that, \(f(\lambda^{4}x, \lambda^{6}y) = \lambda^{k}f(x, y)\) for any \(\lambda \in \mathbb{C}^{*}\). From this description, it is easy to see \(\dim \mathcal{H}^{0}(\mathbb{P}(4, 6), \mathcal{O}(k))\) is the number of monomials \(x^{a}y^{b}\) such that \(4a + 6b = k\), or put in another way, the coefficient of \(p^{k}\) in the power series

\[ \frac{1}{(1 - q^{4})(1 - q^{6})}. \]

\(\square\)

**Proposition 2.9.**

\[ \chi(\mathcal{M}_{1,1}, \frac{1}{1 - qH^{-1}} \frac{1}{1 - q_{1} L_{1}}) = \frac{(1 - q_{1})(1 - q^{4} - q^{6} - q_{1}^{2}q^{6} - q_{1}^{2}q^{8} - q_{1}^{4}q^{8})}{(1 - q^{4})(1 - q^{6})}. \]

**Proof.** By Serre duality, \(H^{1}(\mathbb{P}(4, 6), \mathcal{O}(k)) \simeq H^{0}(\mathbb{P}(4, 6), \mathcal{O}(-10 - k))^{\vee}\), so \(\chi(\mathbb{P}(4, 6), \mathcal{O}(k)) = h^{0}(\mathcal{O}(k)) - h^{0}(\mathcal{O}(-k - 10))\), and the proposition now follows from the previous lemma. \(\square\)
APPENDIX A. A SIMPLE PROOF OF PANDHARIPANDE’S VANISHING THEOREM

The purpose of this appendix is to give a very simple and self-contained proof of Theorem 0.3, first proved in [11]. Recall that the theorem states that at genus zero

\[(A.1) \quad H^j(M_{0,n}, \otimes_{i=1}^n L_i^{d_i}) = 0 \quad \text{for } j \geq 1 \text{ and } d_i \geq 0. \]

We will prove (A.1) by induction on the pair \((n, \sum d_i)\). Note that \(M_{0,3}\) is a point, (A.1) holds obviously for \(n = 3\) with any \(d_i \geq 0\).

When \(n > 3\), we distinguish two cases as the induction methods are different:

1. \(d_i \geq 1\) for all \(i\).
2. One of the \(d_i\) is zero.

For the first case, let \(V := \oplus_{i=1}^n L_i\), choose sections \(s_i\) of \(L_i\) such that the zero locus of the section of \(V\) determined by \(s_i, 1 \leq i \leq n\) is empty. (See the lemma below) Then the Koszul complex

\[
0 \longrightarrow \mathcal{O} \xrightarrow{d} V \xrightarrow{d} \bigwedge^2 V \xrightarrow{d} \cdots \xrightarrow{d} \bigwedge^n V \xrightarrow{0},
\]

is exact, where the differential \(d\) is defined as \(d := \sum_{i=1}^n s_i \wedge\) with \(s_i\) considered as a section of \(V\). Tensoring this complex with \(\otimes_{i=1}^n L_i^{d_i-1}\), we get a resolution of \(\otimes_{i=1}^n L_i^{d_i}\) and \(H^* (\overline{M}_{0,n}, \otimes L_i^{d_i})\) can be computed using hypercohomology.

Consider the double complex \((C^p(U, K^q); \delta, d) \{ p, q \geq 0, q < n\}\), where \(U\) is a covering of \(\overline{M}_{0,n}\), \(K^q := \bigwedge^q V \otimes_{i=1}^n L_i^{d_i-1}\), \(C^p\) are the Čech cochain groups, \(\delta\) is the Čech differential.

\[
\begin{array}{ccc}
\delta & d & \delta \\
\downarrow & & \downarrow \\
c^0(U, \mathcal{O}(V)) & c^1(U, \mathcal{O}(V)) & c^2(U, \mathcal{O}(V)) \\
\delta & d & \delta \\
\downarrow & & \downarrow \\
c^0(U, \mathcal{O}) & c^1(U, \mathcal{O}) & c^2(U, \mathcal{O}) \\
\delta & d & \delta \\
\downarrow & & \downarrow \\
p & q & \end{array}
\]

The method presented in this appendix can also be used to compute \(H^0(\overline{M}_{0,n}, \otimes L_i^{d_i})\). It is also hoped that this method can help to produce an \(S_n\)-equivariant version of our genus zero formula [8], which is needed in the quantum \(K\)-theory [9] computation of general target spaces.
Using two canonical filtrations (by $p$ and $q$ respectively), we obtain two spectral sequences $'E_r^{p,q}$ and $"E_r^{p,q}$ with

$$
'\ E_1^{p,q} = H^p(\mathcal{M}_{0,n}, \mathcal{K}^q),
$$

$$
"\ E_2^{p,q} = H^p(\mathcal{M}_{0,n}, \mathcal{H}_d^d(K^*)).
$$

These two spectral sequences abut to the same hyper-cohomology $H^*(\mathcal{M}_{0,n}, K^*)$.

By induction, $'E_1^{p,q} = 0$ if $p \neq 0$, since $\mathcal{K}^q$ is the direct sum of $\otimes L_i^d$'s with $\sum d_i' < \sum d_i$. So $'E_r^{p,q}$ degenerates at $r=1$, and $H^q(\mathcal{M}_{0,n}, K^*) = 0$ when $q \geq n$, as $'E_1^{p,q} = 0, q \geq n$ by our construction.

Note that $"E_2^{p,q}$ is zero for $q \neq n-1$, and $"E_2^{p,n-1} = H^p(\mathcal{M}_{0,n}, \mathcal{O} \otimes L_i^d)$, $"E_r^{p,q}$ degenerates at $r=2$. Therefore, $H^p(\mathcal{M}_{0,n}, \mathcal{O} \otimes L_i^d) = H^{p+n-1}(\mathcal{M}_{0,n}, K^*) = 0$ when $p + n - 1 \geq n$, or $p \geq 1$.

**Lemma A.1.** A generic section of the vector bundle $\bigoplus_{i=1}^{n-2} L_i$ on $\mathcal{M}_{0,n}$ has empty zero locus. Here $n > 3$, and generic means outside a proper closed subset of the affine space $H^0(\mathcal{M}_{0,n}, \bigoplus_{i=1}^{n-2} L_i)$.

**Proof.** The lemma can be proved by induction. Note that having empty zero locus is an open property for sections, the statement in the lemma is equivalent to the existence of a section having empty locus.

It is easy to see the lemma holds for $n = 4$ by identifying $\mathcal{M}_{0,4}$ with $\mathbb{P}^1$, and $L_1, L_2$ with $\mathcal{O}_{\mathbb{P}^1}(1)$.

For $n > 4$, consider the forgetful map $\pi : \mathcal{M}_{0,n} \to \mathcal{M}_{0,n-1}$. Since $L_i = \pi^* L_i(D_i), 1 \leq i \leq n-2$, where $D_i$ is the image of the $i$-th section of $\pi$, a section $t_i$ of $L_i$ on $\mathcal{M}_{0,n-1}$ would induce a section $s_i$ of $L_i$ on $\mathcal{M}_{0,n}$ with support $\text{Supp } s_i = \pi^{-1}(\text{Supp } t_i) \cup D_i$. It is straightforward to check $\cap_{i=1}^{n-2} \text{Supp } t_i = \emptyset$ if for all $1 \leq j \leq n-2, \cap_{i=1}^{n-2} \text{Supp } t_i = \emptyset$, and these conditions hold for generic $t_i$ by induction. \hfill $\square$

For the other case, assume $d_n = 0$ and consider the forgetful map $\pi : \mathcal{M}_{0,n} \to \mathcal{M}_{0,n-1}$. As $H^1(C, \mathcal{O}_C(D)) = 0$ for $C$ rational and degree of $\mathcal{O}_C(D)$ positive, $R^1\pi_* (\otimes_{i=1}^{n-1} L_i^{d_i}) = 0$ by cohomology and base change. Then we have a degenerated Leray spectral sequence which gives

$$
\begin{align*}
H^l(\mathcal{M}_{0,n}, \otimes_{i=1}^{n-1} L_i^{d_i}) &= H^l(\mathcal{M}_{0,n-1}, R^0\pi_* (\otimes_{i=1}^{n-1} L_i^{d_i})) \\
&= H^l(\mathcal{M}_{0,n-1}, (\otimes_{i=1}^{n-1} L_i^{d_i}) \otimes (\mathcal{O} + \sum_{i_d \neq 0} \sum_{m=1}^{d_i} L_i^{-m})).
\end{align*}
$$
and this is zero by induction. Here we used the string equation (Prop. 1.8) that $K$-theoretically

$$\pi_*(\otimes_{i=1}^{n-1} L_i^{d_i}) = (\otimes_{i=1}^{n-1} L_i^{d_i}) \otimes (\mathcal{O} + \sum_{i,d_i \neq 0} \sum_{m=1}^{d_i} \otimes L_i^{-m}).$$

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