Mapping Class Groups
MSRI, Fall 2007
Day 8, October 25

November 26, 2007

Reducible mapping classes Review terminology:

- An *essential curve* γ on S is a simple closed curve γ such that:
  - no component of $S - \gamma$ is a disc with 0 or 1 puncture
- An *essential curve system* Γ on S is a pairwise disjoint union of essential curves.
  - Components of Γ are allowed to be isotopic.
  - Γ is *pairwise nonisotopic* if no two components are isotopic.
- An *essential surface* in S is a subsurface-with-boundary $F \subset S$ such that:
  - F is a closed subset
  - ∂F is an essential curve system
  - No two components of F are isotopic.

  It is possible for an annulus component of F to be isotopic into one or two other components.

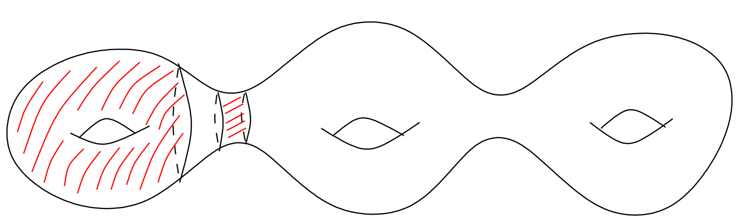

Figure 1: This is an example of an essential surface. The boundary of the subsurface may contain isotopic curves, but no two componants of the subsurface are isotopic.

- $\mathcal{MCG}(S)$ acts on isotopy classes of
  - essential curves;
  - essential curve systems;
  - essential subsurfaces
- $\phi \in \mathcal{MCG}(S)$ is *reducible* if any of the following equivalent conditions are satisfied:
  - $\exists$ a (pairwise nonisotopic) essential curve system Γ whose isotopy class is invariant under ϕ.
  - Γ is a *reduction system* for ϕ.
  - Can choose a representative Φ of ϕ such that $\Phi(\Gamma) = \Gamma$.
  - Can furthermore choose Φ so that $\Phi(N(\Gamma)) = N(\Gamma)$.
- For each component F of $\overline{S - N(\Gamma)}$ (We think of the interior of F as a punctured surface):
  - Least $n \geq 1$ such that $\Phi^n(F) = F$ is the *first return time* of F.
  - The mapping class of Φ^n is a *component mapping class* of F.

Well-definedness of component mapping classes

- Slight problem:
  - Γ only well-defined up to isotopy,
  - $\overline{S - N(\Gamma)}$ only well-defined up to isotopy,
- Action of ϕ on set of isotopy classes of components of Γ (and of $N(\Gamma)$) is well-defined
- Action of ϕ on set of isotopy classes of components of $\overline{S - N(\Gamma)}$ is well-defined.
- First return times are well-defined.
- First return mapping classes are well-defined up to "conjugacy-by-isotopy".

Canonical reduction system

- Reduction systems for ϕ need not be unique (up to isotopy):
  - A component mapping class might be reducible $\implies$ the reduction system can be enlarged.
  - A reduction system might have more than one orbit under action of ϕ $\implies$ the reduction system can be shrunk (see figure).

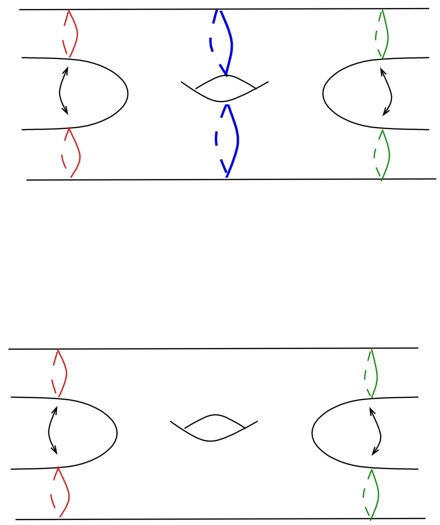

Figure 2: If on the depicted surface the map is a hyperelliptic involution then we can remove the two thick blue curves to get a smaller reducing system.

Theorem 1. *For each $\phi \in \mathcal{MCG}(S)$ there is a reduction system Γ, possibly empty, which is uniquely characterized up to isotopy by the following:*

1. *For each reduction system Γ' for ϕ, each $\gamma \in \Gamma$, and each $\gamma' \in \Gamma'$, $\langle \gamma, \gamma' \rangle = 0$*

2. *Γ is maximal with respect to previous property.*

Γ *is* also *uniquely characterized up to isotopy by the following:*

3. *Each component mapping class is either finite order or irreducible - in fact, pseudo-Anosov.*

4. *Γ is minimal with respect to the previous property.*

Corollary: $\Gamma \neq \emptyset$ if and only if ϕ has infinite order and is not pseudo-Anosov.

Remarks

- Original source of (3) and (4) is hard to pin down...
  appears in print in more than one place...probably first known to Thurston.

- (1) and (2) is in paper of Handel–Thurston.

- (1) and (2) is an analogue of the JSJ decomposition of a 3-manifold.

- The analogy is (no coincidence?) very strong: the canonical JSJ decomposition of the mapping torus of ϕ corresponds exactly to the canonical reduction system of ϕ.

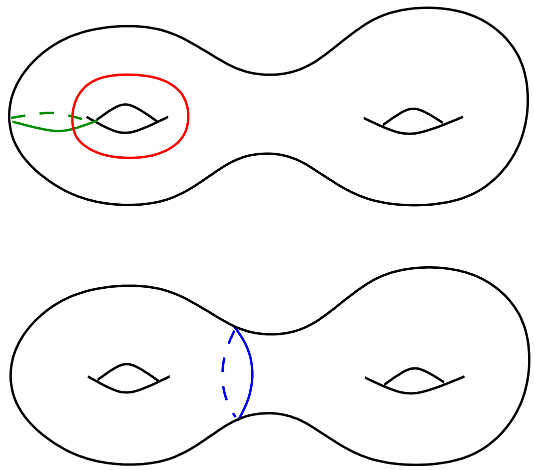

Figure 3: If $\gamma_1 \in \Gamma$ intersects $\gamma_1' \in \Gamma'$ then can find a curve $\beta_1 = \partial N(\gamma_1 \cup \gamma_1')$ which is disjoint from both and is still a reducing curve, and the automorphism restricted to $N(\gamma_1 \cup \gamma_2)$ is periodic. Can continue this way to cut up the surface into periodic/pseudo-Anosov pieces.

Reducible conjugacy invariants: The reduction graph

Let $\phi \in \mathcal{MCG}(S)$ be reducible and of infinite order. Define:
$\Gamma_\phi = \{\gamma_i \mid i = 1, \ldots, I\}$ = canonical reduction system.
$\{F_j \mid j = 1, \ldots, J\}$ = components of $S - \Gamma_\phi$.
$\mathcal{G}_\phi$ = *reduction graph*:

- Vertex V_j for each component F_j of $\overline{S - N(\Gamma)}$,
  - labelled with the integer genus(F_j)
  - (valence of V_j will equal $|\partial F_j|$)
- Edge E_i for each component γ_i of Γ

Let $\mathrm{Stab}(\Gamma) = \{\theta \in \mathcal{MCG} | \theta(\Gamma) = \Gamma\}$.

Lemma 2. *The labelled isomorphism type of $\mathcal{G}_\phi$ is a conjugacy invariant of ϕ.* $\diamondsuit$

Given reducible $\phi, \phi' \in \mathcal{MCG}(S)$, *assume* $\mathcal{G}_\phi$, $\mathcal{G}_{\phi'}$ are isomorphic as labelled graphs.

- Choose an isomorphism $\mathcal{G}_\phi \mapsto \mathcal{G}_{\phi'}$.
- Lift it to $\psi \in \mathcal{MCG}(S)$ taking Γ_ϕ to $\Gamma_{\phi'}$ (by the classification of surfaces)
- Both ϕ and $\phi'' = \psi^{-1}\phi'\psi$ are in the subgroup

$$\mathrm{Stab}(\Gamma_\phi) < \mathcal{MCG}(S)$$

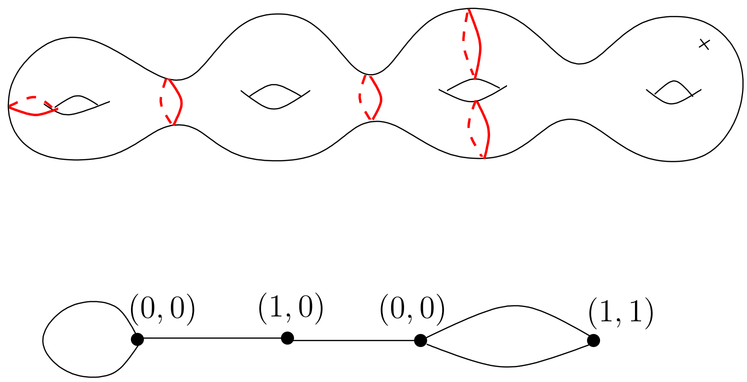

Figure 4: To a reduction system we associate a labeled graph with labels (genus,punctures).

- TFAE:
    1. ϕ, ϕ' are conjugate in $\mathcal{MCG}(S)$
    2. ϕ, ϕ'' are conjugate in $\mathcal{MCG}(S)$
    3. ϕ, ϕ'' are conjugate in $\mathrm{Stab}(\Gamma_\phi)$
- Proof of 2 $\implies$ 3: Recall that $\Gamma_{\phi''} = \Gamma_\phi$. If $\theta\phi\theta^{-1} = \phi''$ then $\theta\Gamma_\phi = \Gamma_{\phi''} = \Gamma_\phi$ so $\theta \in \mathrm{Stab}(\Gamma_\phi)$.

Structure of $\mathrm{Stab}(\Gamma)$

Given essential curve system $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$, let $F_1, \ldots, F_k$ be components of $S - N(\Gamma)$.

1. $\mathrm{Stab}(\Gamma)$ acts on $\mathcal{G}_\Gamma$. Let $\mathrm{Stab}_0(\Gamma)$ be the kernel:

$$1 \to \mathrm{Stab}_0(\Gamma) \to \mathrm{Stab}(\Gamma) \to (\text{finite group}) \to 1$$

2. $\mathrm{Stab}_0(\Gamma)$ acts (up to isotopy) on each F_i:

$$1 \to T(\Gamma) \to \mathrm{Stab}_0(\Gamma) \to \prod_{i=1}^{k} \mathcal{MCG}_0(F_i) \to 1$$

   where $\mathcal{MCG}_0$ means punctures are fixed.

3. $T(\Gamma)$ is a free abelian group having as basis the Dehn twists

$$\tau_{\gamma_1}, \ldots, \tau_{\gamma_m}$$

Each of these three items yields reducibility invariants.

1. Action on $\mathcal{G}_\phi$.

- ϕ has well-defined action on $\mathcal{G}_\phi$, preserving genus labels.
- Follows from:
  - uniqueness of Γ,
  - well-definedness up to isotopy of action on Γ
  - and on $\overline{S - N(\Gamma)}$.

Fact. *The conjugacy type of the ϕ action on $\mathcal{G}_\phi$ is a conjugacy invariant.* $\diamondsuit$

2. Action on components F_i of $S - N(\Gamma_\phi)$.

- For each vertex V_F, augment its label with the conjugacy invariants of the first return mapping class of F.

Fact. $\mathcal{G}_\phi$*, with genus labels, action, and augmented labels, is a conjugacy invariant of* ϕ*.*

BUT, there is still hidden conjugacy information in the action on the components . . .

Pairing punctures

- Each $\gamma \in \Gamma$ corresponds to TWO punctures of $S - N(\Gamma)$.
- Each puncture has a certain “identity” in the conjugacy invariants for the first return mapping classes.
- The pairing of punctures, or more strictly speaking the pairing of their “identities” in the conjugacy invariants of first return mapping classes, is itself a conjugacy invariant.
- Formally:
  - Let D be the collection of unordered pairs of punctures of $S - N(\Gamma)$, one pair for each $\gamma \in \Gamma$.
  - All pseudo-Anosov and finite order conjugacy classes, on the components of $S - N(\gamma)$, need to be lifted to conjugacy invariants in the group $\mathcal{MCG}(S - N(\gamma), D)$.

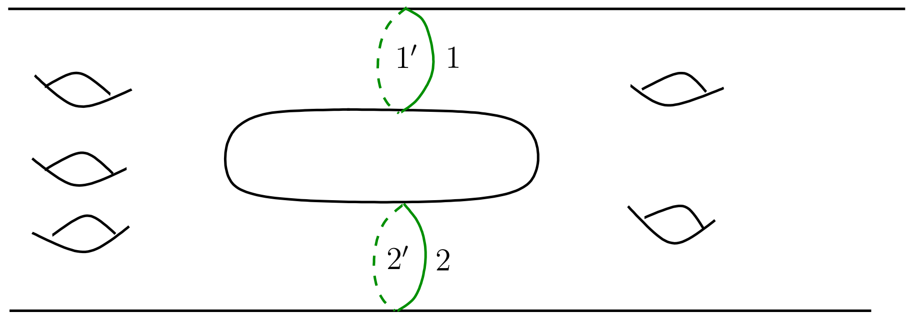

Figure 5: So far, the conjugacy invariants will not distinguish between to mapping classes which are conjugate on each subsurface seperately (but not as homeomorphisms of the whole surface). Pairing the punctures gets rid of this discrepancy.

- Extra finite amount of bookkeeping: add finite amount of data to pseudo-Anosov and to finite order conjugacy invariants ...
- Call this the "puncture gluing" invariants.

Fact. *$\mathcal{G}_\phi$, with previous action and labels, and with added information of puncture gluing invariants, is a conjugacy invariant.*

3. Twist invariants So far we can't distinguish a Dehn twist from its square, up to conjugacy.

- To each $\gamma \in \Gamma$ we shall associate a *twist invariant*, a rational number

$$r(\gamma) \in \mathbf{Q}$$

- For example, given a Dehn twist power τ_γ^k we will have $r(\gamma) = k$.
- Note: $\partial\overline{S - N(\gamma)} = \partial \bigcup_{j=1}^{J} F_j$
- On each component c of each F_j there is a natural periodic set:
  - All of c (if F_j is finite order)
  - Endpoints of stable and unstable leaves
    (if F_j is pseudo-Anosov)
- Choose product structure $N(\gamma_i) = S^1 \times [0, 1]$
- Require action of Φ on $S^1 \times 0$ and $S^1 \times 1$ to be either:
  - Rigid rotation (on finite order boundary)
  - Alternating source-sink with evenly spaced periodic points (on pseudo-Anosov boundary)

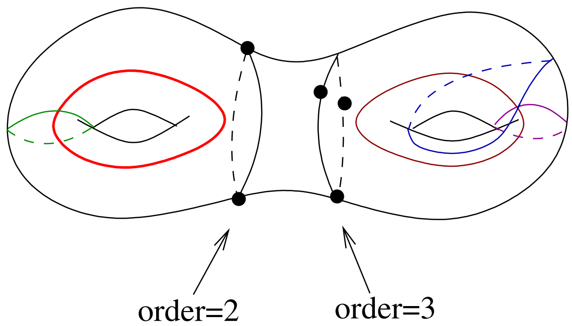

Figure 6: The order of the mapping class on the left is 4 and on the right 6. Thus the orders of their conjugacy classes are 2 and 3. This yields a twist of a multiple of $\frac{1}{6}$ in the middle curve.

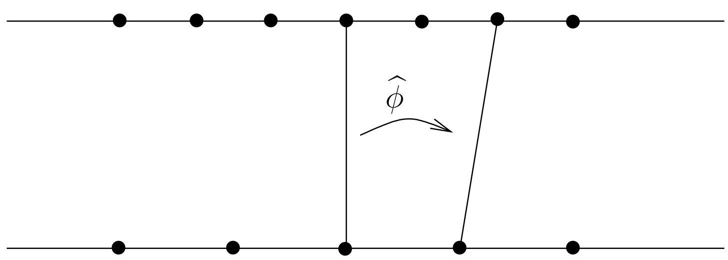

Figure 7: To find the twist parameter - lift to the universal cover and find the reciprocal of the slope of the image of a straight line.

- Lift (first return of) Φ to universal covering map

$$\mathbf{R} \times [0,1] \quad \text{of} \quad S^1 \times [0,1] = N(\gamma_i)$$

- Define the twist to be $r(\gamma) = \frac{1}{\text{slope of } \widetilde{\Phi}(0 \times [0,1])}$

Theorem 3. *The following data gives a complete conjugacy invariant of a reducible $\phi \in \mathcal{MCG}(S)$:*

- *Reduction graph $\mathcal{G}_\phi$*
- *Label for each vertex: genus*
- *Label for each vertex: conjugacy invariant of first return*
- *Label for each edge: puncture pairing data*
- *Label for each edge: twist*
- *Action of ϕ on $\mathcal{G}_\phi$*

Computing the conjugacy invariant

- Bestvina-Handel paper gives an algorithm for finding either:
  - Invariant train track
  - Reducing system
- Even in the finite order case, the algorithm will produce an invariant graph.
- When the algorithm finds a reducing system Γ, *don't stop*:
  - Obtain a graph having the reducing system Γ as a subgraph.
  - Continue the algorithm, relative to Γ.
- Continue inductively through smaller and smaller subsurfaces
- In the end, on each subsurface, have either train track or invariant graph.
- Pseudo-Anosov invariants are computable from train track
- Finite order invariants are computable from invariant graphs
- Puncture pairing data: evident from incidence of reducing curves, train tracks, invariant graphs.
- Also have: Extra branch coming off the side of each reducing curve.
- Twist: can compute using extra branches.