

Examples of Conformal Maps and of Critical Points

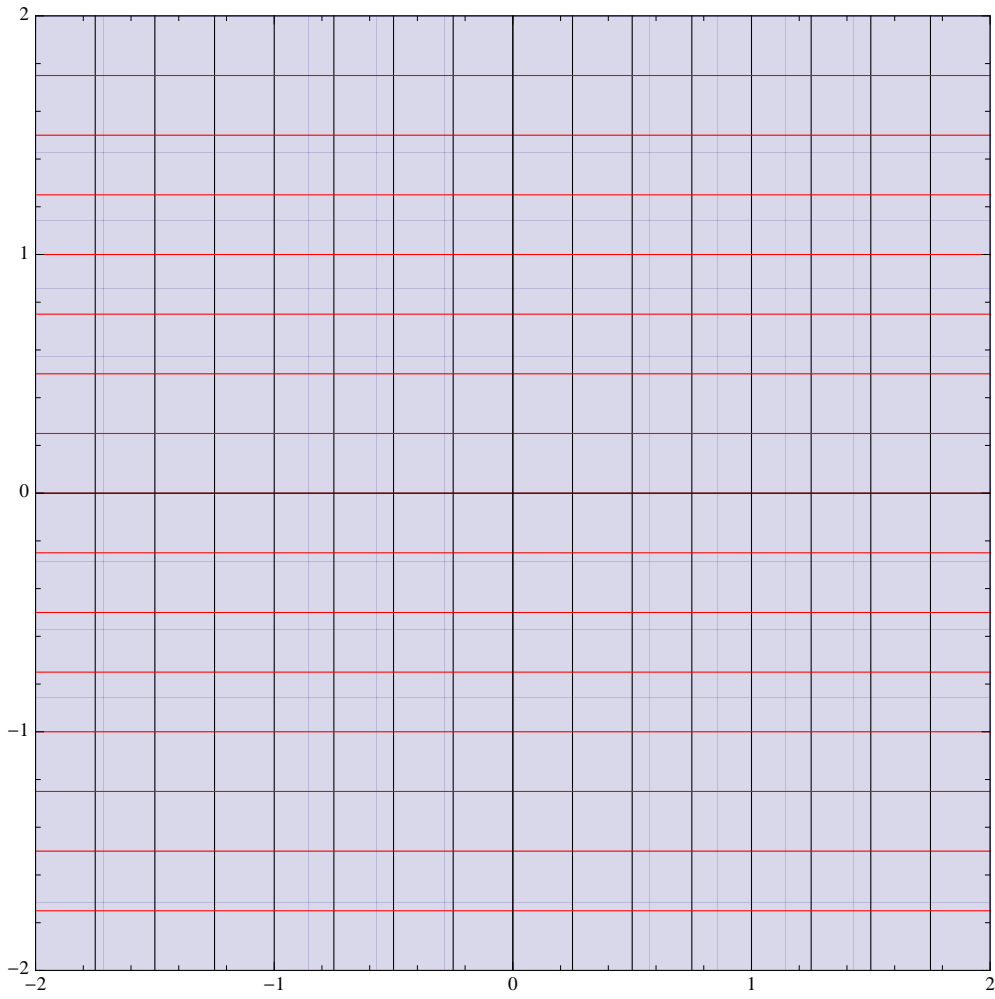
We know that an analytic function $f(z)$ is conformal (preserves angles and orientation) at all points where the derivative $f'(z)$ is not zero. Here we look at some examples of analytic functions that illustrate that they are conformal maps. They also show what happens at places where $f'(z) = 0$.

To understand the pictures, we first draw the standard mesh of horizontal and vertical lines in the complex plane, then we plot the resulting mesh under the function $f(z)$. The curves in this mesh meet at right angles, which is consistent with conformal. To know that the map is conformal, we also need to know that the curves in the mesh are moving at the same speed at any given point of intersection.

In the pictures we will also see what happens at the Critical Points. These are, by definition, the points where $f'(z) = 0$.

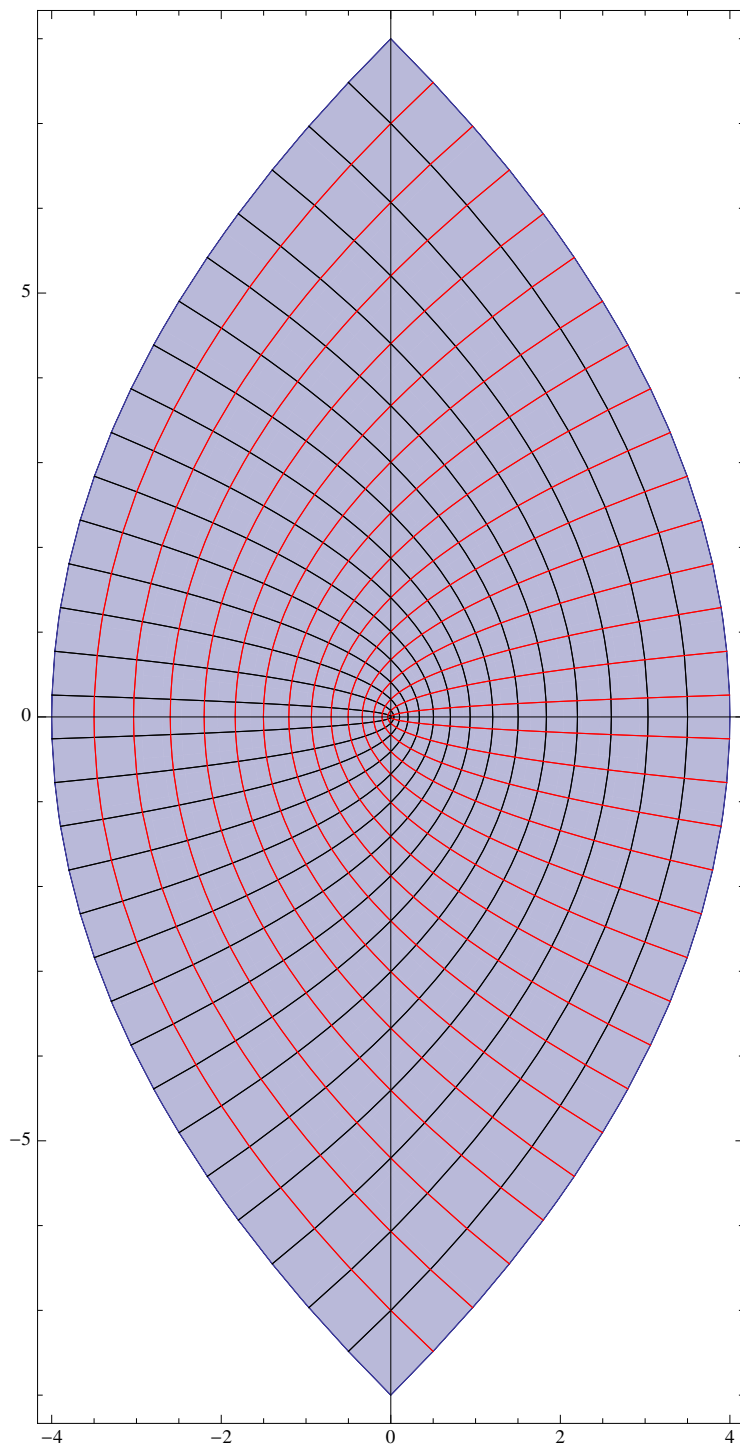
To begin, draw the standard mesh in the z-plane:

```
ParametricPlot[{Re[(u + i v)^1], Im[(u + i v)^1]},  
  {u, -2, 2}, {v, -2, 2}, MeshStyle -> {Black, Red}, PlotRange -> 2]
```



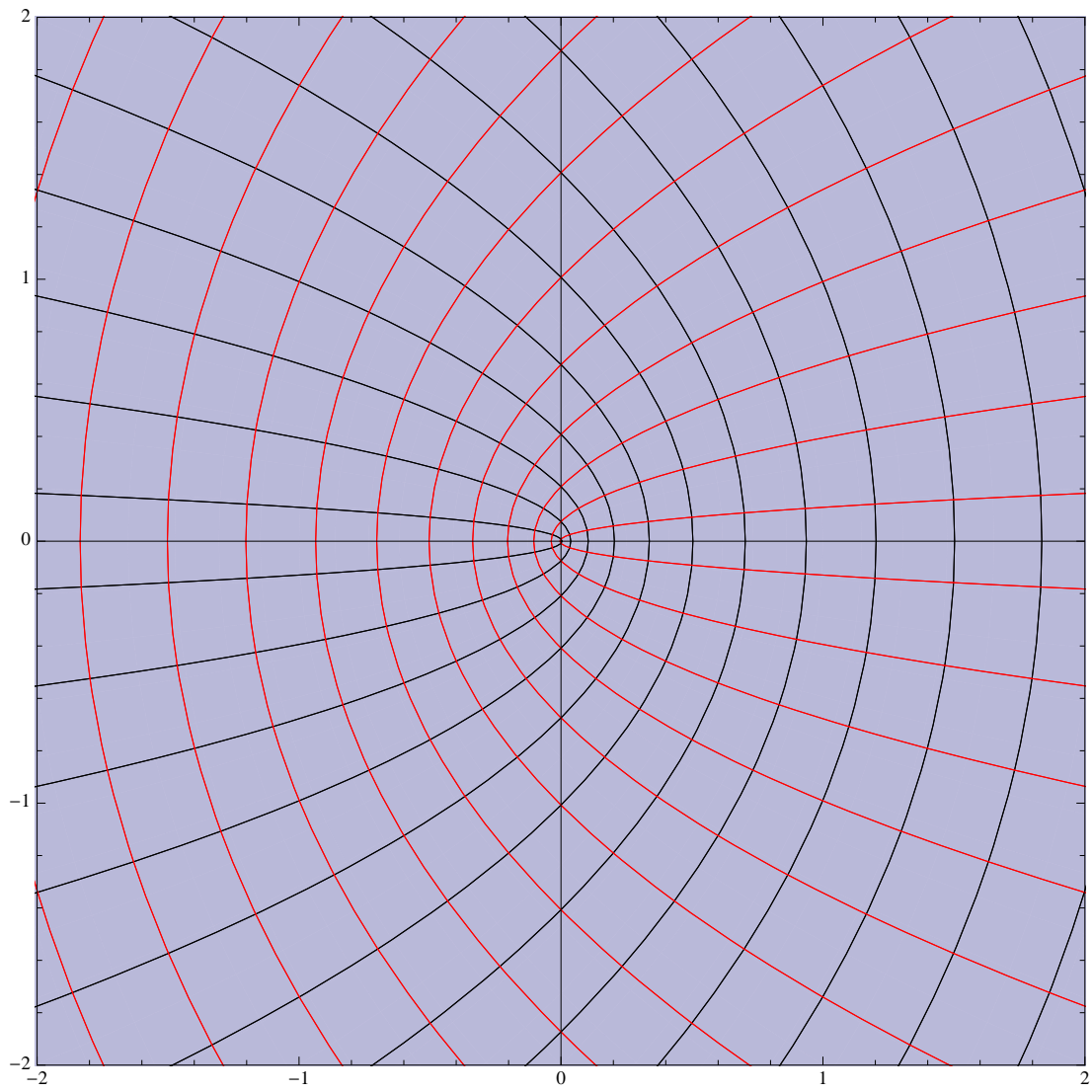
As a first example, draw its image under the squaring function $z \rightarrow z^2$:

```
ParametricPlot[{Re[(u + i v)^2], Im[(u + i v)^2]},
  {u, -2, 2}, {v, -2, 2}, Mesh -> 30, MeshStyle -> {Black, Red}]
```



Observe that they indeed meet at right angles, except at the critical point $z = 0$. Here's another view:

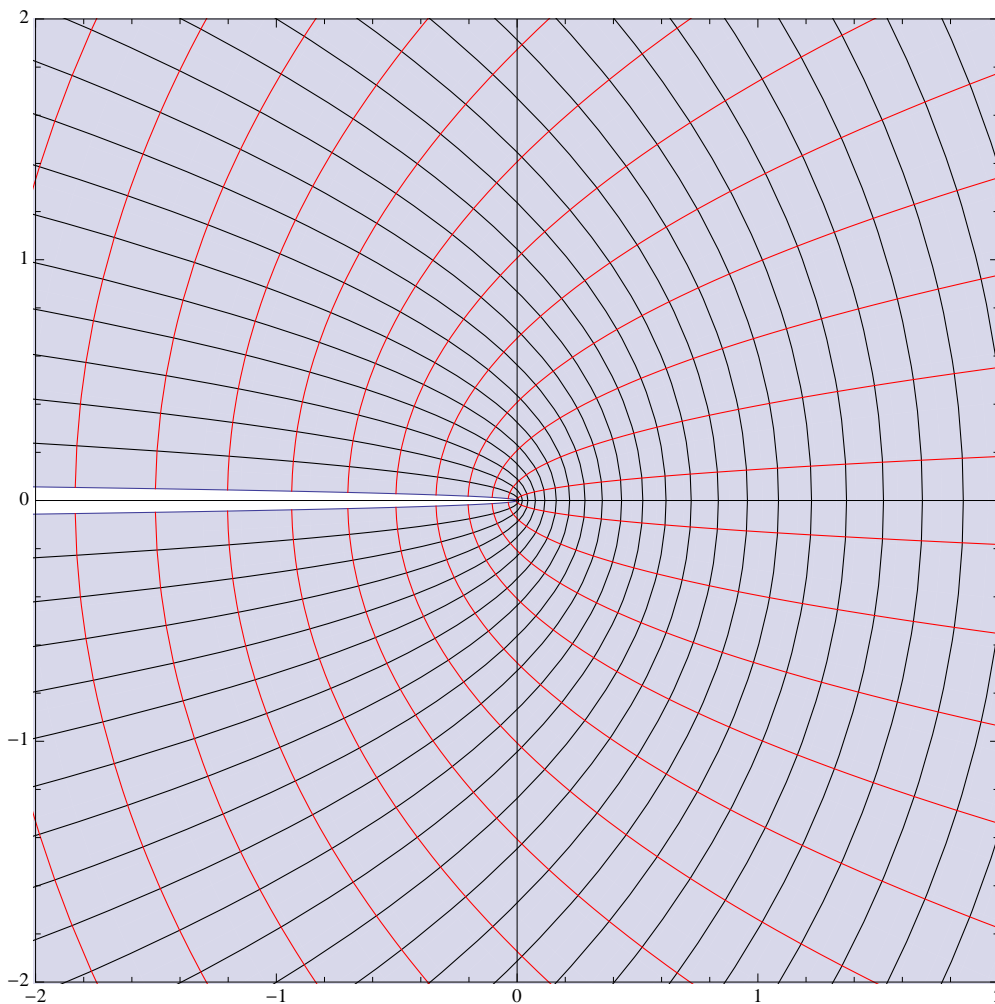
```
ParametricPlot[{Re[(u + i v)^2], Im[(u + i v)^2]}, {u, -2, 2},  
{v, -2, 2}, Mesh -> 30, MeshStyle -> {Black, Red}, PlotRange -> 2]
```



Observe what happens at the critical point. The red curves bunch up on the positive real axis, the black ones on the negative real axis. This is because the real axis maps 2:1 on the positive real axis, while the imaginary axis maps 2:1 on the negative real axis. So the 90 degree angle at the origin between the real and imaginary axes gets stretched to a 180 degree angle.

We know that $z \rightarrow z^2$ is 2:1, it cannot be inverted. But it is 1:1 when restricted to the open half-plane bound by any line through 0, in particular the imaginary axis. The following illustrates the image of the half-plane $\text{Re}(z) > 0$, which is the plane slit along the negative real axis:

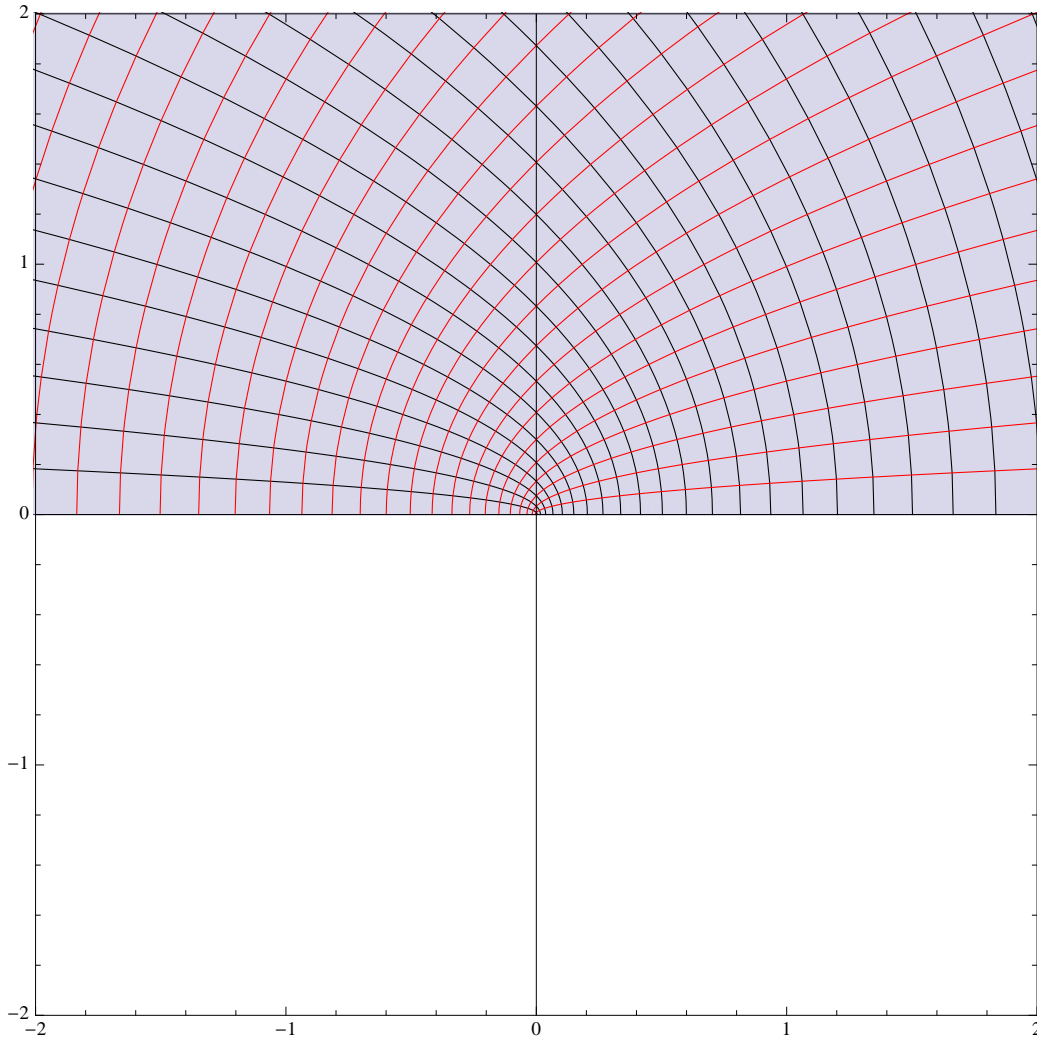
```
ParametricPlot[{Re[(u + i v)^2], Im[(u + i v)^2]}, {u, .02, 2},
  {v, -2, 2}, Mesh -> 30, MeshStyle -> {Black, Red}, PlotRange -> 2]
```



So this slit plane serves as the domain of a branch of the square root function.

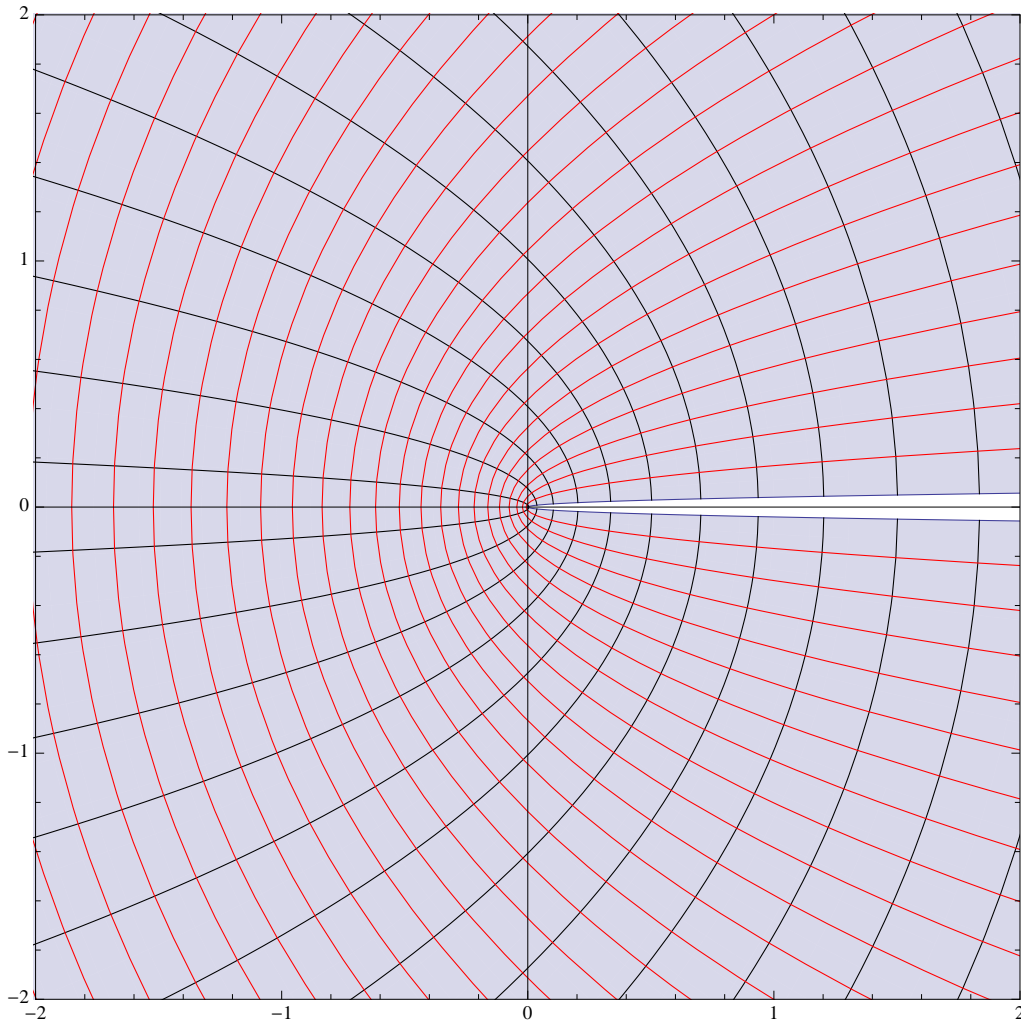
Next, here's the image of the first quadrant. Note again how the 90 degree angle at the origin (the critical point) becomes a 180 degree angle.

```
ParametricPlot[{Re[(u + i v)^2], Im[(u + i v)^2]}, {u, 0, 2},  
  {v, 0, 2}, Mesh -> 30, MeshStyle -> {Black, Red}, PlotRange -> 2]
```



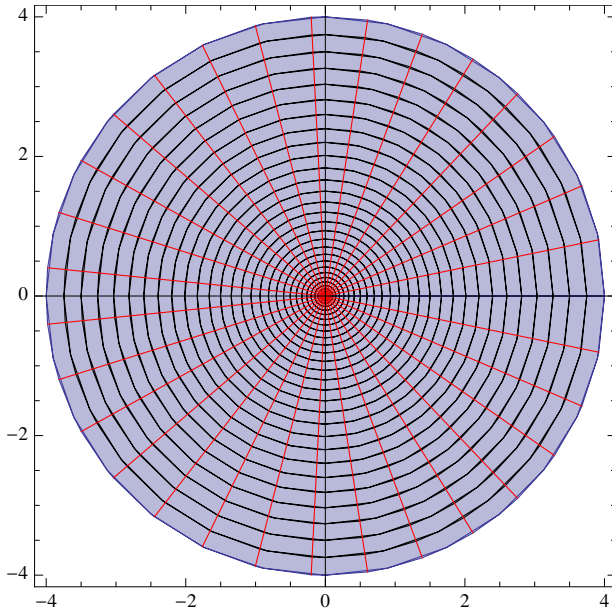
Finally the image of the upper-half plane $\text{Im}(z) > 0$. This is the plane slit along the positive real axis. It would be the the domain of another branch of the square root, one not defined on positive real numbers.

```
ParametricPlot[{Re[(u + i v)^2], Im[(u + i v)^2]}, {u, -2, 2},
  {v, .02, 2}, Mesh -> 30, MeshStyle -> {Black, Red}, PlotRange -> 2]
```

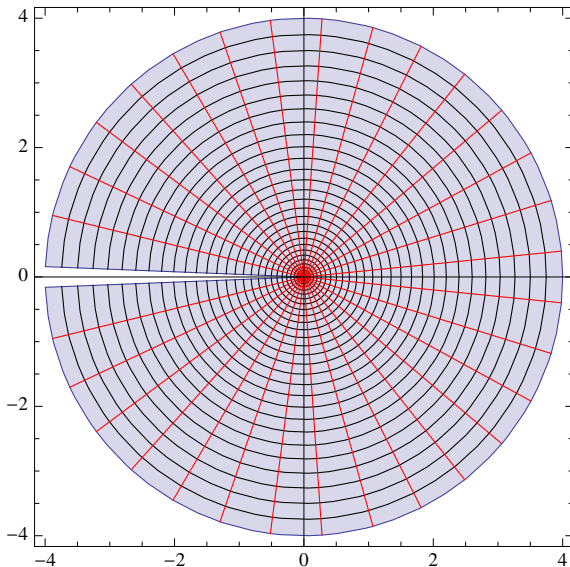


We can also draw pictures in polar coordinates, the first indicating that $z \rightarrow z^2$ is 2:1, the second showing the image of $-\pi/2 < \arg(z) < \pi/2$, the same as the half-plane $\operatorname{Re}(z) > 0$.

```
ParametricPlot[{Re[(u Exp[I v])^2], Im[(u Exp[I v])^2]},
  {u, 0, 2}, {v, 0, 2 Pi}, Mesh -> 30, MeshStyle -> {Black, Red}]
```

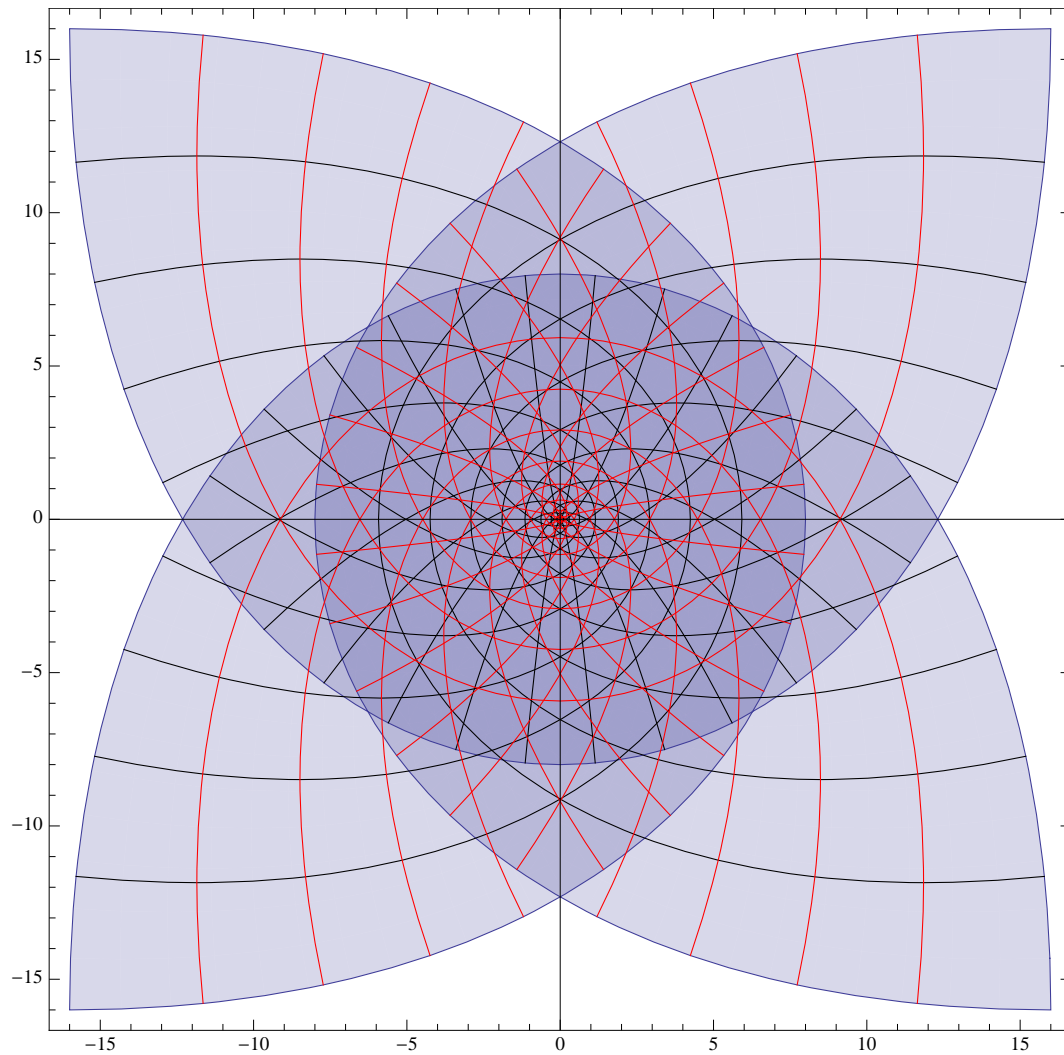


```
ParametricPlot[{Re[(u Exp[I v])^2], Im[(u Exp[I v])^2]}, {u, 0, 2},
  {v, -Pi/2 + .02, Pi/2 - .02}, Mesh -> 30, MeshStyle -> {Black, Red}]
```



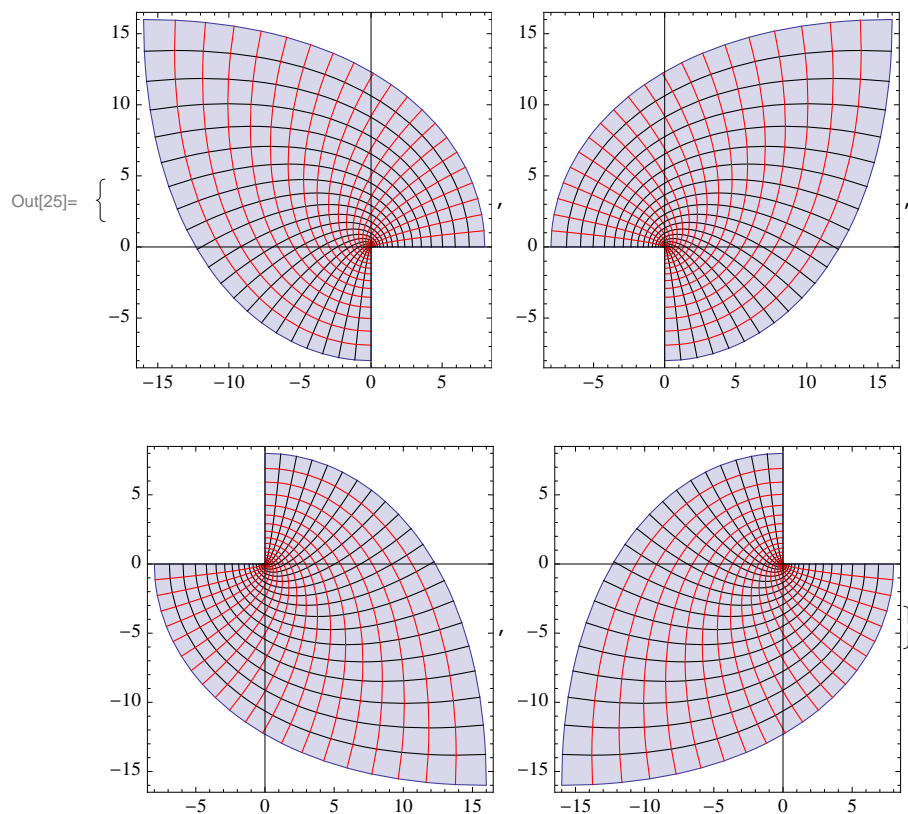
Now to $z \rightarrow z^3$, which should be more complicated than $z \rightarrow z^2$ since now not only $f'(0) = 0$ but in addition the second derivative $f''(0) = 0$.

```
ParametricPlot[{Re[(u + i v)^3], Im[(u + i v)^3]},  
{u, -2, 2}, {v, -2, 2}, Mesh -> 20, MeshStyle -> {Black, Red}]
```



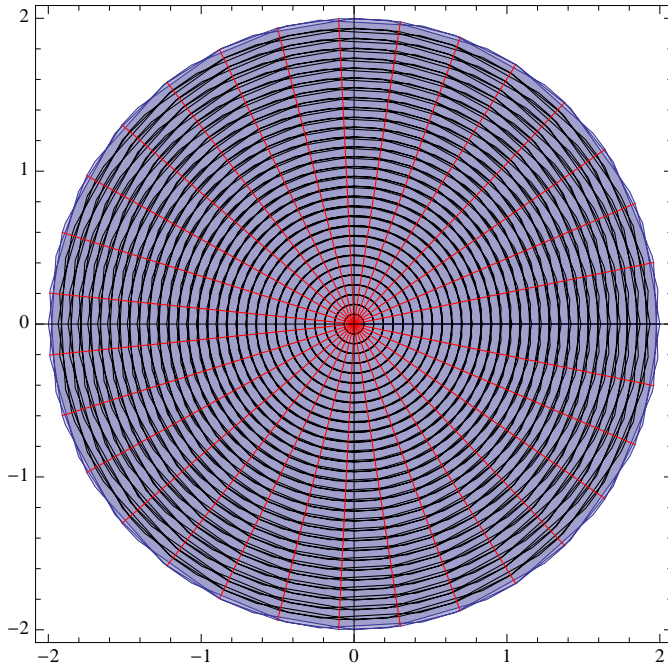
To understand this picture, look at the image of each quadrant. At the critical point at the origin the 90 degree angle gets multiplied by 3. The next four pictures give the image of each quadrant. You could piece them together to see how they fit into the original picture. Start at a point on the positive real axis in the first picture, travel counterclockwise in a circle around the origin until you reach the negative imaginary axis, then jump to the second picture, and so on, until you get back to the same point on the positive real axis on the fourth picture.

```
In[25]:= {ParametricPlot[{Re[(u + i v)^3], Im[(u + i v)^3]}, {u, 0, 2}, {v, 0, 2}, Mesh -> 20,
  MeshStyle -> {Black, Red}], ParametricPlot[{Re[(u + i v)^3], Im[(u + i v)^3]},
  {u, -2, 0}, {v, 0, 2}, Mesh -> 20, MeshStyle -> {Black, Red}],
  ParametricPlot[{Re[(u + i v)^3], Im[(u + i v)^3]}, {u, -2, 0}, {v, -2, 0}, Mesh -> 20,
  MeshStyle -> {Black, Red}], ParametricPlot[{Re[(u + i v)^3], Im[(u + i v)^3]},
  {u, 0, 2}, {v, -2, 0}, Mesh -> 20, MeshStyle -> {Black, Red}}]}
```



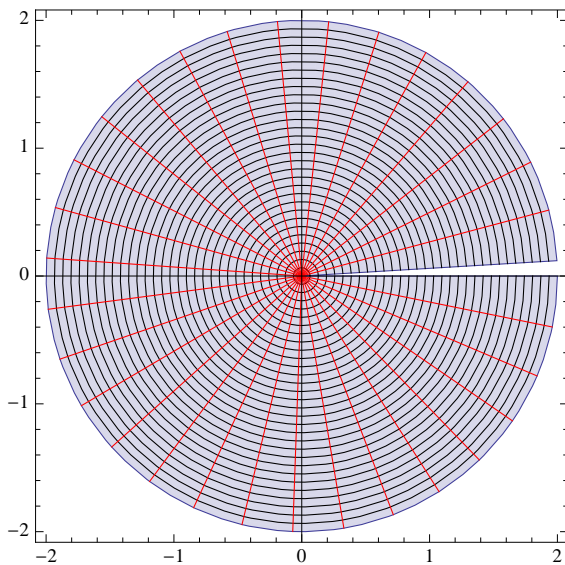
Of course it may be more natural to use polar coordinates. First we look at how the whole plane triply covers its image:

```
ParametricPlot[{Re[(u Exp[3 I v])], Im[(u Exp[3 I v])]},  
{u, 0, 2}, {v, 0, 2 Pi}, Mesh -> 30, MeshStyle -> {Black, Red}]
```



Then the image of a 120 degree sector, which is a slit plane, one possible domain for a branch of the cube root function.

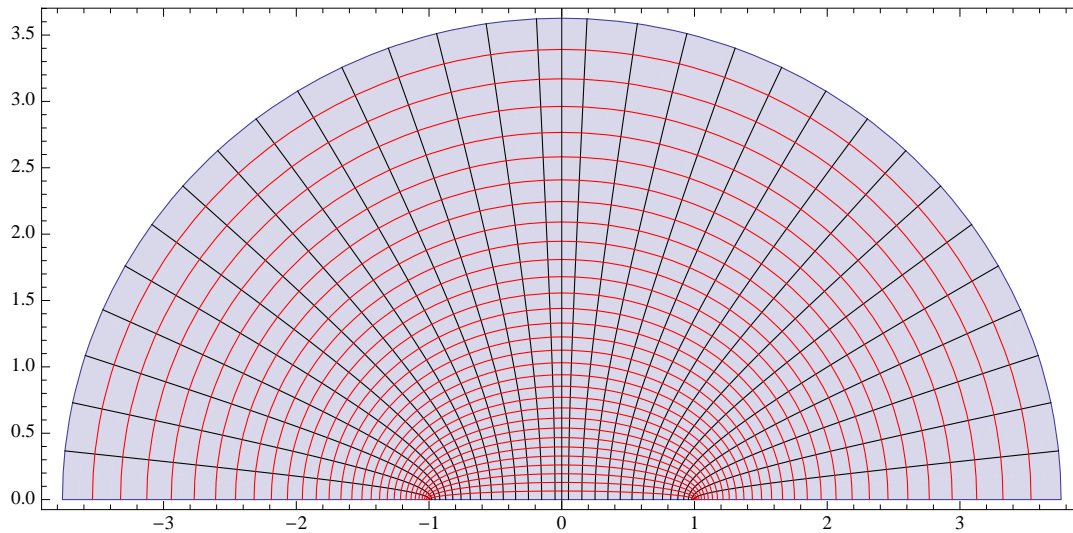
```
ParametricPlot[{Re[(u Exp[3 I v])], Im[(u Exp[3 I v])]},  
{u, 0, 2}, {v, 0.02, 2 Pi / 3}, Mesh -> 30, MeshStyle -> {Black, Red}]
```



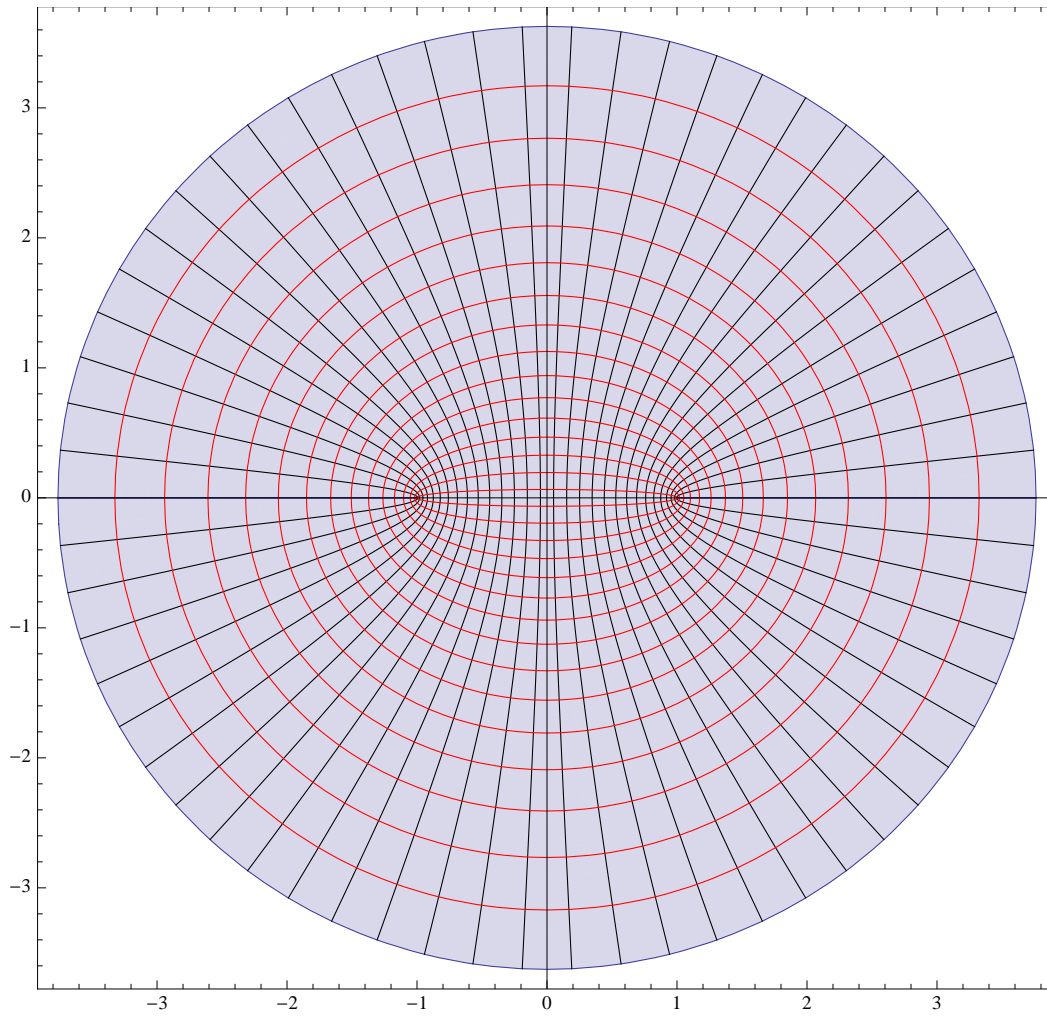
Sin(z)

Now we look at $\sin(z)$. The first picture shows the image of the half-strip where $-\pi/2 < \operatorname{Re}(z) < \pi/2$ and $\operatorname{Im}(z) > 0$. The second as the image of the whole strip. Images of boundaries are included.

```
ParametricPlot[{Re[Sin[u + i v]], Im[Sin[u + i v]]},  
{u, -Pi/2, Pi/2}, {v, 0, 2}, Mesh -> 30, MeshStyle -> {Black, Red}]
```



```
ParametricPlot[{Re[Sin[(u + i v)]], Im[Sin[(u + i v)]]},  
{u, -Pi / 2, Pi / 2}, {v, -2, 2}, Mesh -> 30, MeshStyle -> {Black, Red}]
```

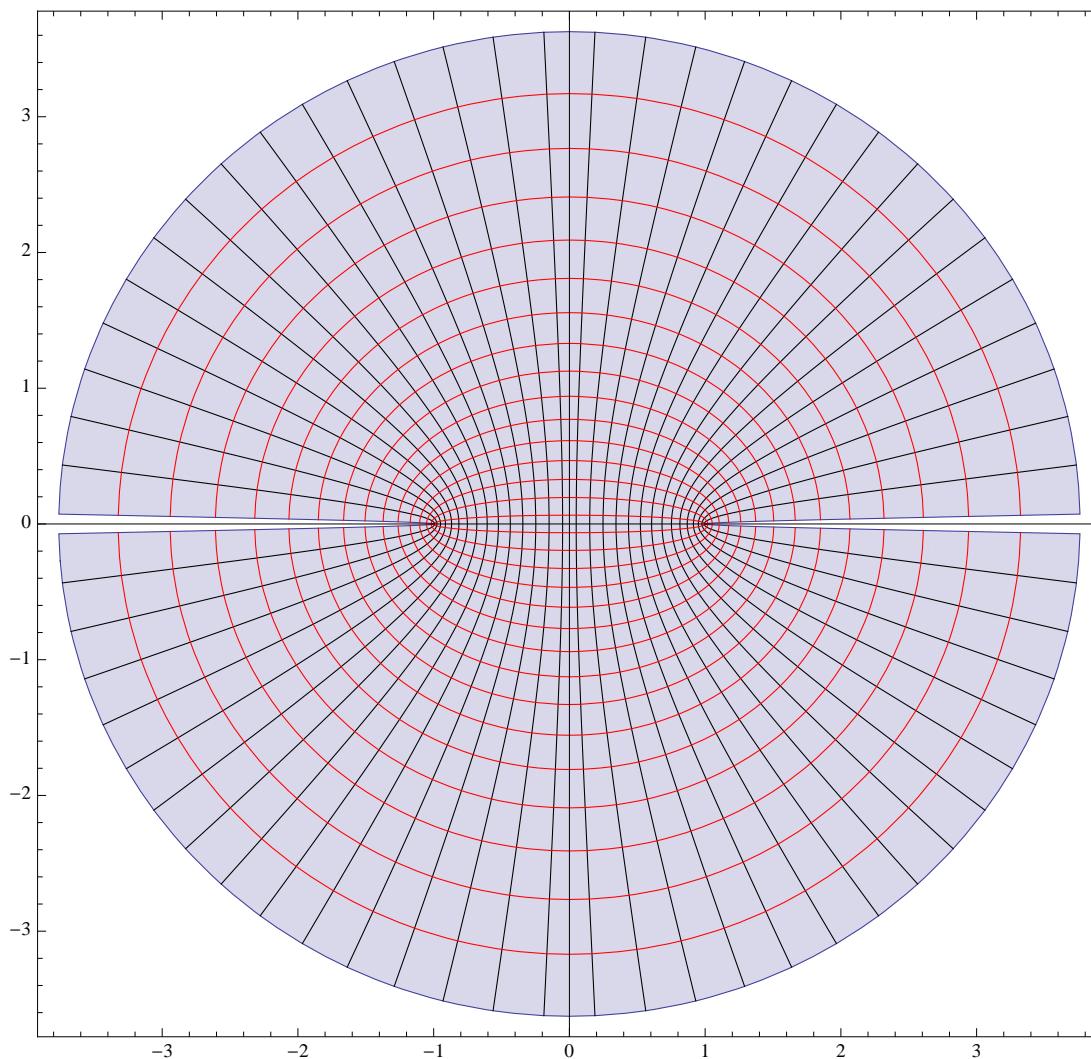


Observe what we already know: images of horizontal lines are ellipses, collapsing to the interval $[-1, 1]$ on the real axis, and the images of the vertical lines are hyperbolas, collapsing to the interval $[1, \infty)$ as they approach 1, and to $(-\infty, -1]$ as they approach -1.

Observe that $\pi/2, -\pi/2$ are critical points with images 1, -1. Look at the pictures near 1 and -1 and compare them with the pictures of z^2 at the origin. The similarity is not a coincidence. The derivatives $\sin'(\pi/2) = \sin'(-\pi/2) = 0$, but the second derivatives $\sin''(\pi/2), \sin''(-\pi/2)$ are not zero. This means that the quadratic approximation to $\sin(z)$ dictates the behavior near these critical points, in other words, they should look roughly like (constant) z^2 .

Finally, if you take the interior of the above strip, it maps 1:1 onto its image, which is a doubly slit plane. So this is a possible domain for arcsin.

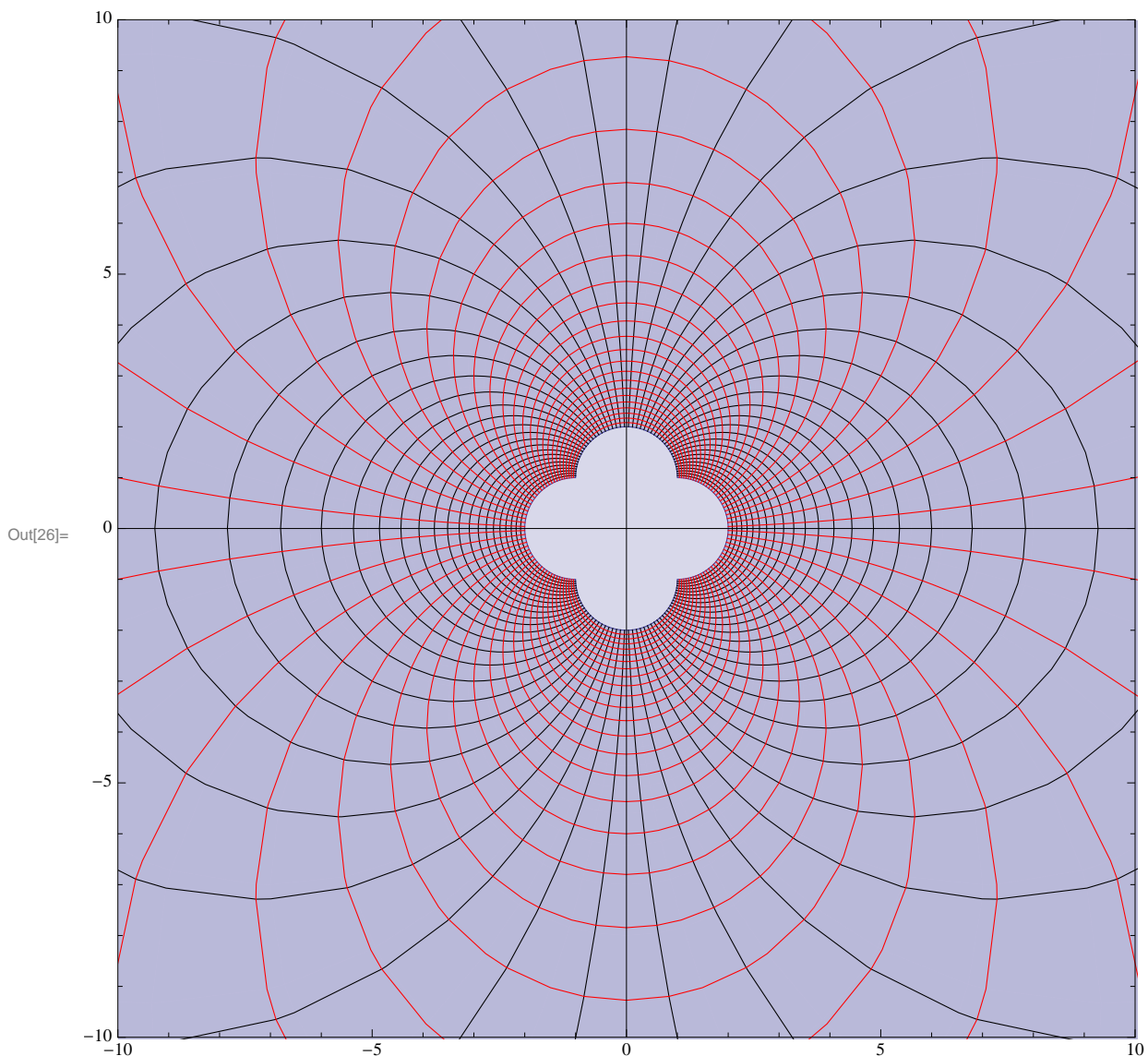
```
ParametricPlot[{Re[Sin[u + i v]], Im[Sin[u + i v]]},
  {u, (-Pi/2) + .02, (Pi/2) - .02}, {v, -2, 2}, Mesh -> 30, MeshStyle -> {Black, Red}]
```



$1/z$ and $(z+1)/(z-1)$

These two transformations appear in the homework. Here are some pictures. First a picture of $1/z$ for z in the box $|\operatorname{Re}(z)| < 1/2$, $|\operatorname{Im}(z)| < 1/2$. The gap you see in the middle is the image of the complement of this box, illustrating that this function sends infinity to 0. If you were to keep expanding the values allowed in the image, you'd see that the picture never stops, illustrating that 0 goes to infinity.

```
In[26]:= ParametricPlot[{Re[1 / (u + i v)], Im[1 / (u + i v)]}, {u, -.5, .5},
  {v, -.5, .5}, Mesh -> 50, MeshStyle -> {Black, Red}, PlotRange -> 10]
```



Finally a picture of what the function $(z + 1) / (z - 1)$ does to a box around the point $z = 1$ which goes to infinity. The gap you see about the point 1 is the image of the

complement of this box, illustrating that infinity goes to 1.

```
In[27]:= ParametricPlot[{Re[(u + 1 + I v) / (u - 1 + I v)], Im[(u + 1 + I v) / (u - 1 + I v)]},  
  {u, -2, 4}, {v, -4, 4}, Mesh -> 70, MeshStyle -> {Black, Red}, PlotRange -> 5]
```

