# NOTES IN ALGEBRAIC TOPOLOGY
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## 1. INTRODUCTION

These are notes for a course in characteristic classes. These notes are meant to be rather informal, skipping many details, but trying to convey the main ideas, and to give some appreciation for the rich nature of the subject. The standard reference is the book [2]. This book deals mostly with the topological theory, is the foundation of the subject. But there is also the differential geometric theory, which we hope to develop. The characteristic classes appear then in de Rham cohomology, and [1] gives a very clear exposition of de Rham theory. Algebraic geometry provides many applications for the theory of characteristic classes. We hope to be able to discuss, to some extent, all these points of view.

*Warning*: These are informal notes, sloppy in many details, sometimes deliberate. Some examples to keep in mind:

(1) Coefficients in homology and cohomology groups are often left unstated. Will often write $H^k(M)$ rather than $H^k(M,\mathbb{Z})$ or $H^k(M,\mathbb{R})$.
(2) Assumptions on the topological spaces involved are not made very explicit. Usually a simplicial complex will do, or manifold if we're usiing differentiial forms.
(3) Sometimes we are careful to distinguish a simplicial complex $K$ (a combinatorial object) from its geometric realization $|K|$ (a topological space), sometimes we are sloppy and make no distinction.
(4) Many equations hold only up to sign. For instance, a factor $(-1)^{pq}$ may be missing.

1.1. **The first example.** Characteristic Classes are certain invariants of vector bundles that allow us to distinguish and, to a certain extent, classify bundles. They should have some geometric significance, tell us something interesting about the bundles. They should be *functorial* and *computable*.

The first example of a "characteristic class" is the invariant that distinguishes the Möbius band from the cylinder:

**Proposition 1.** *Let $E \xrightarrow{p} S^1$ be a reai line bundle over $S^1$ (vector bundle with fiber $\mathbb{R}$). Then*

$$E \cong \Big([0,1]\times\mathbb{R}\Big) \Big/ \Big((0,y)\sim(1,ay)\Big) \textit{ for some } a\in\mathbb{R}, a\neq 0.$$

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*and $a/|a| \in \{-1, 1\}$ is a complete invariant of $E$. In particular, there are exactly two isomorphism classes of real line bundles over $S^1$: the cylinder $E_1$ and the Möbius band $E_{=1}$.*

*Proof.* Clear from the fact that if $q : [0,1] \to S^1$ is the quotient map $q(t) = \exp(2\pi i t)$, then $q^*E$ is trivial. $\square$

We will see that it's best to interpret $\{\pm 1\}$ as $H^1(S^1, \mathbb{Z}/2)$. The invariant $a/|a|$ will be denoted $w_1(E)$ called the *first Stiefel-Whitney class* of $E$.

Here is a hint for why we should use $H^1(S^1, \mathbb{Z}/2)$ as our two-element set of complete invariants: If we cosider a more general topological space $X$ and want to classify real line-bundles $E \to X$, it is reasonable to expect that the bundles $f^*E$ over $S^1$, for the various maps $f : S^1 \to X$, may be involved in the classification. So an invariant $w_1(E) \in H^1(X, \mathbb{Z}/2)$ with the functoriality property $w_1(f^*(E)) = f^*(w_1(E)) \in H^1(S^1, \mathbb{Z})$ would make sense. This is indeed the case, and this more general $w_1$ will be the first Stiefel-Whitney class.

1.2. **Vector Bundles over Spheres.** The reasoning behind the classification of line bundles over $S^1$ applies to vector bundles over $S^m$ with fiber $\mathbb{R}^n$ for any $m, n \geq 1$, see [3]. Namely, suppose $E \to S^m$ is a vector bundle with fiber $\mathbb{R}^n$. Write $S^m$ as

$$S^m = D^m_+ \sqcup_{S^{m-1}} D^m_-,$$

a union of two disks (hemispheres) glued along their common equatorial sphere $S^{m-1}$. Since disks are contractible, we have trivializations

$$\phi_\pm : E|_{D^m_\pm} \xrightarrow{\cong} D^m_\pm \times \mathbb{R}^n,$$

and over their common equator $S^{m-1}$ a bundle isomorphism

$$\psi = \phi_+ \circ \phi_-^{-1} : S^{m-1} \times \mathbb{R}^n \to S^{m-1} \times \mathbb{R}^n$$

which is of the form

$$\psi(x, v) = (x, A(x)v) \text{ for some } A : S^{m-1} \to GL(n, \mathbb{R}), \tag{1}$$

where $GL(n, \mathbb{R})$ denotes the group of invertible $n$ by $n$ - matrices.

Conversely, given a continuous map $A : S^{m-1} \to GL(n, \mathbb{R})$ we can form the identification space $E_A$ defined as

$$E_A = D^m_+ \times \mathbb{R}^n \bigsqcup_{(x,v)\sim(x,A(x)v)} D^m_- \times \mathbb{R}^n \tag{2}$$

and with $p : E_A \to S^m$ defined by $p(x, v) = x$ on each piece, gives us a bundle isomorphic to $E$. Thus all bundles over $S^m$ arise in this way. Moreover the isomorphism class of $E_A$ depends just on the homotopy class of $A$, that is, $[A] \in \pi_{m-1}(GL(n, \mathbb{R})$. In other words, if we introduce the temporary notation $Vec_n(S^m)$ for the collection of vector bundles of rank $n$ over $S^m$ and $[Vec_n(S^m)]$ for the set of isomorphism classes of these bundles. Then we have:

**Proposition 2.** *The map $\pi_{m-1}(GL(n, \mathbb{R})) \to [Vec_n(S^m)]$ that assigns to $[A] \in \pi_{m-1}(GL(n, \mathbb{R}))$ the class $[E_A] \in Vec_n(S^m)$ is a one-to-one correspondence.*

To apply this proposition, it is convenient to replace $GL(n,\mathbb{R})$ by its subgroup $O(n)$, the orthogonal group, since the inclusion $O(n)\to GL(n,\mathbb{R})$ is a homotopy equivalence. Also recall that $O(n)$ has two connected components, its identity component is the special orthogonal group $SO(n)$. Also recall that for $i>0$, $\pi_i(O(n))=\pi_i(SO(n))$, the identity component, So in Proposition 2 we can replace $\pi_{m-1}(GL(n,\mathbb{R}))$ by $\pi_0(O(n))$ if $m=1$ and by $\pi_{m-1}(SO(n))$ if $m\geq 2$.

Finally we need to know the values $\pi_{m-1}(O(n))$. Let's look at small values of $m$:

**Proposition 3.** (1) $\pi_0(O(n))=\mathbb{Z}/2$ *for all* $n\geq 1$

(2) $\pi_1(SO(n))=\begin{cases}\mathbb{Z} & \text{if } n=2\\ \mathbb{Z}/2 & \text{if } n>2.\end{cases}$

(3) $\pi_2(SO(n))=0$ *for all* $n$*;*

(4) $\pi_3(SO(n))=\begin{cases}\mathbb{Z} & \text{if } n=3 \text{ or } n>4\\ \mathbb{Z}\oplus\mathbb{Z} & \text{if } n=4.\end{cases}$

*Proof.* For (1): If is well-known that $O(n)$ has two connecred components, classified by the determinant $\pm 1$, thus $\mathbb{Z}/2$ in additive notation. Therefore $\pi_0(O(n))=\mathbb{Z}/2$ for all $n\geq 1$.

For all the remaining ones can replace $O(n)$ by $SO(n)$.

For the first part of (2), $SO(2)$ is the rotation group of $\mathbb{R}^2$, homeomorohic to the circle $S^1$, thus $\pi_1(SO(2))=\mathbb{Z}$.

For the second part of (2), let us first look at $SO(3)$, the group of rotations of $S^2$, hence two ways to proceed:

(1) Identify $SO(3)$ as a topological space: $SO(3)=\mathbb{R}P^3$ the real projective space,
(2) Use the exact homotopy sequence of the fibration $SO(2)\to SO(3)\to S^2$:

$$\to \pi_2(S^2)\xrightarrow{\partial_*}\pi_1(SO(2))\to\pi_1(SO(3))\to\pi_1(S^2))\to$$

which becomes $\mathbb{Z}\xrightarrow{\partial_*}\mathbb{Z}\to\pi_1(SO(3))\to 0$ where $\partial_*$ is the *connecting homomorphism* of the exact homotopy sequence of the fibration. So, for this approach, it is critical to compute the connecting homomorphism.

Leaving the connecting homomorphism aside for now, let's accept $SO(3)=\mathbb{R}P^3$ and hence $\pi_1(SO(3))=\mathbb{Z}/2$.

To complete (2), need to compute $\pi_1(SO(n)$ for $n>3$, Again the homotopy sequence of the fibration $SO(n-1)\to SO(n)\to S^{n-1}$ :

$$\to\pi_2(S^{n-1})\to\pi_1(SO(n-1))\to\pi_1(SO(n))\to\pi_1(S^{n-1})\to$$

which becomes $0\to\pi_1(SO(n-1))\to\pi_1(SO(n))\to 0$ that is, $\pi_1(SO(n-1))=\pi_1(SO(n)$ for $n>3$. Therefore $\pi_1(SO(n))=\mathbb{Z}/2$ for all $n\geq 3$. This finishes the proof of (2) for all $n$. But it also establishes another important fact:

**Proposition 4.** *The inclusion* $SO(n)\to SO(n+1)$ *induces an isomorphism* $\pi_i(SO(n))\xrightarrow{\cong}\pi_i(SO(n+1))$ *for* $i\leq n-2$.

*Proof.* Look at the exact homotopy sequence of $SO(n)) \to SO(n+1) \to S^n$:

$$\to \pi_{i+1}(S^n) \to \pi_i(SO(n)) \to \pi_i(SO(n+1)) \to \pi_i(S^n) \to$$

if both $i+1$ and $i$ are both $< n$, that is, $i \leq n-2$, this sequence becomes

$$0 \to \pi_i(SO(n)) \to \pi_i(SO(n+1)) \to 0$$

which proves the claim. □

Finally,for statements (3) and (4), if we accept three facts: $SO(3) = \mathbb{R}P^3$ as above, equivalently, the group $Q$ of unit quaternions (homeomorphic to the sphere $S^3$ ) double-covers $SO(3))$ and fhe group $Q \times Q$ doublle - covers $SO(4)$ (by the map $Q \times Q \to SO(4)$ that takes $(q_1, q_2)$ to the linear tranformation of $\mathbb{H} = \mathbb{R}^4$ given by $x \to q_1 x q_2^{-1}$) it follows that $\pi_2(SO(3)) = \pi_2(SO(4)) = 0$ and, by Proposition 4, $\pi_2(SO(n)) = 0$ for all $n$. Finally $\pi_3(SO(3)) = \mathbb{Z}$ and $\pi_3(SO(4)) = \mathbb{Z} \oplus \mathbb{Z}$ follow, while $\pi_3(SO(5))$ will take some work (a connecting homomorphism), after that $\pi_3(SO(n)), n > 5$ follows from Proposition 4. □

If we apply these facts to vector bundles over $S^m$ we get the following:

(1) $\mathbb{R}^n$-bundles over $S^1$ in one-one correspondence with $\mathbb{Z}/2 = H^1(S^1, \mathbb{Z}/2)$ by the invariant $w_1(E)$ called the *first Stiefel-Whitney class*.
(2) $\mathbb{R}^2$-bundles over $S^2$ in bijective correspondence with $\mathbb{Z}$, the invariant in $H^2(S^2, \mathbb{Z}) = \mathbb{Z}$ called the *Euler class* $e(E)$.
(3) Instead of $\mathbb{R}^2$-bundles we could look at $\mathbb{C}$ - bundles, in this case the multiplicative group of $\mathbb{C}$ is homotopy equivalent to $U(1)$, isomorphic to $SO(2)$, we get again a bijective correspondence to $\mathbb{Z}$, the invariant in $H^2(S^2, \mathbb{Z})$ is called the *first Chern class* $c_1(E)$.
(4) $\mathbb{R}^n$-bundles over $S^2$ in one-one correspondence with $\mathbb{Z}/2 = H^2(S^2, \mathbb{Z}/2)$, the invariant $w_2(E)$ is called the *second Stiefel-Whitney class*.
(5) $\mathbb{R}^n$-bundles over $S^3$: all trivial.
(6) $\mathbb{R}^4$-bundles over $S^4$: in one-one correspondence with $\mathbb{Z}^2$ (pairs of integers), in corresponcence the pair $(e(E), p_1(E))$, the *Euler class* and *first Pontrjagin class*.
(7) $\mathbb{R}^n$-bundles over $S^4$, $n \geq 5$: one invariant in $\mathbb{Z} = H^4(S^4, \mathbb{Z})$, the *first Pontrjagin class* $p_1(E)$.

## 2. Euler class

One of the earliest theorems in characteristic classes is the Poincaré - Hopf theorem:

**Theorem 1.** *Let $M$ be a closed manifold of dimension $n$ and $X$ a continuous vector field on $M$ with isolated zeros. Then*

$$\sum_{X_p=0} \iota(p) = \chi(M).$$

Here $\chi(M) = \sum(-1)^i \dim(H^i(M, \mathbb{R}))$ is the *Euler characteristic* or *Euler number* of $M$. If $p$ is an isolated zero of $X$, in some local coordinate system defined on a neighborhood $U$ centered at $p$, $X_q = (q, f(q))$ for some map $f : (U, U \setminus p) \to (\mathbb{R}^n, \mathbb{R}^n \setminus 0)$, then $\iota(p) = \deg(f)$.

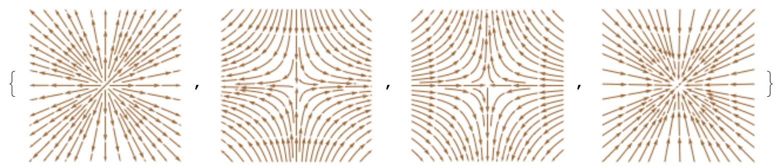

FIGURE 1. Zeros of index $1, -1, -1, 1$ respectively

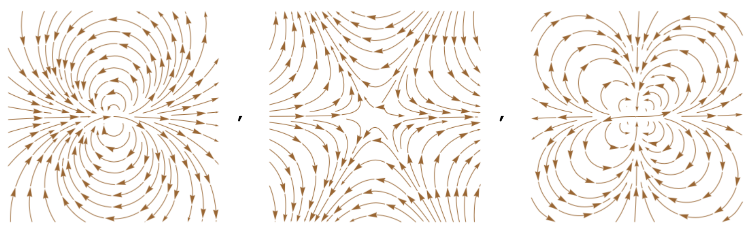

FIGURE 2. Zeros of index $2, -2, , 3$ respectively

Hopf proved this theorem in two steps:

(1) $\sum_{X(p)=0} \iota(p)$ is independent of $X$, provided $X$ has isolated zeros.

(2) If $M$ is triangulated by a simplicial complex $K$, that is, $M$ is homeomoprhic to $|K|$, the geometric realization of $K$, construct a vector field in the first barycentric subdivison $|K'|$ that vanishes prescisely at the barycenters $\hat{\sigma}$ of the simplices of $K$ with index $\iota(\hat{\sigma}) = (-1)^{\dim(\sigma)}$

Then for the field $X$ of (2) we get

$$\sum_{X(p)=0} \iota(p) = \sum_{i=0}^{n} (-1)^i (\text{number of } i - \text{simplices of } K) = \chi(M)$$

where the second equality is a well-known lemma in linear algebra.

See Figure 3 for a picture of Hopf's vector field for the first barycentric subdivision of a square (or , with some care, of a cell structure on a torus). Another standard and easy way to justify the second step is to use the gradient $\nabla f$ of a Morse function $f : M \to \mathbb{R}$ and the well known equality $\chi(M) = \sum_{\nabla_p f = 0} \iota(p)$.

To prove (1), let's put it into the more general context of the *Euler class* of an oriented vector bundle. For concreteness we will assume that all spacesand maps are smoothj, but we could use more general topological spaces and continuous maps.

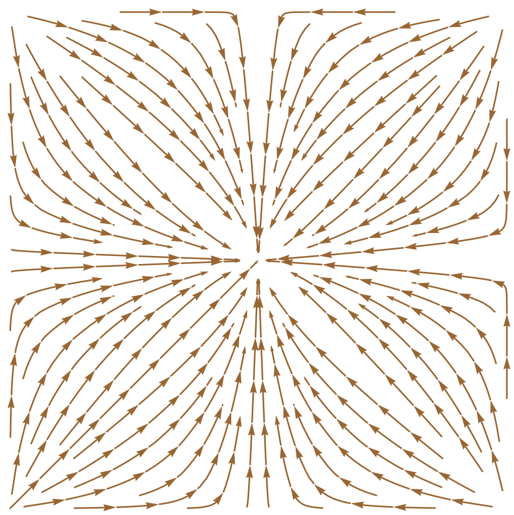

FIGURE 3. Hopf's field on a square (or torus)

2.1. **Orientation.** Suppose $\pi : E \to M$ is a smooth vector bundle with fiber $\mathbb{R}^n$ over a smooth manifold $M$ of dimension $m$. Recall that his means that $M$ has a cover $\mathcal{U} = \{U_\alpha\}$ so that

(1) For each $\alpha$ there is a bundle isomorphism $\phi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^n$.
(2) Whenever $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \emptyset$ the map $\phi_\alpha\phi_\beta^{-1} : U_{\alpha\beta} \times \mathbb{R}^n \to U_{\alpha\beta} \times \mathbb{R}^n$ is of the form $\phi_\alpha\phi_\beta^{-1}(x, v) = (x, a_{\alpha\beta}v)$ for some smooth map $a_{\alpha\beta} : U_{\alpha\beta} \to GL(n, \mathbb{R})$.

Recall that $GL(n, \mathbb{R})$ denotes the group of invertible $n$ br $n$ matrices with real coeffcients, and $GL^+(n, \mathbb{R})$ denotes the subgroup of matrices with positive determinant. It has index two in $GL(n, \mathbb{R})$

An *orientation* on an $n$-dimensional vector space can be defined in two equivalent ways:

(1) A choice of "positive half-space" in the one-dimensional vector space $\Lambda^n V$. Namely, $\Lambda^n V \setminus 0$ has two connected components, choose one and call it *positive*.
(2) A choice of generator for the infinite cyclic group $H^n(V, V \setminus 0, \mathbb{Z})$

    One way to see the equivalence is to show that each is equivalent to a third statement:

(3) Let $e_1, \dots, e_n$ be a basis of $V$.The set $\mathcal{B}$ of ordered bases $\{(e_{\sigma(1)}, \dots, e_{\sigma(n)}) : \sigma \in S_n\}$ is a disjoint union $\mathcal{B} = \mathcal{B}^1 \sqcup \mathcal{B}^{-1}$ according to the sign of the permutation $\sigma$. Choose one of these two sets and call it positive.

We next prove that a choice in (3) gives choices in (1) and (2). Once this is done the opposite directions should be clear. Choose a positive ordered basis in (3) and denote it simply $e_1, \dots, e_n$. Then the positive multiples of $e_1 \wedge \cdots \wedge e_n$ give a component of $\Lambda^n V \setminus 0$, hence an orientation in the sense of (1), Let $\Delta_n$ be the standard $n$-simplex $\{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1, \dots, x_n \geq 0$ and $x_1 + \cdots + x_n \leq 1\}$. Then the map $\sigma : \Delta_n \to V$ defined by $\sigma(x_1, \dots, x_n) = (x_1 e_1 + \cdots + x_n e_n) - (e_1 + \cdots + e_n)/n$ is a singular simplex in $V$ with booundary in $V \setminus 0$ (the reason for the last term as to have the barycenter at the origen). It is easy to see that this is a generator $x$ of the integral homology group $H_n(V, V \setminus 0, \mathbb{Z})$. We then choose the generator of $H^n(V, V \setminus 0, \mathbb{Z})$ that evaluates to one on $x$.

Finally, recall that if $A; V \to V$ is a linear transformation, then the induced linear map $\Lambda^n A : \Lambda^n V \to \Lambda^V$ is multiplication by $\det A$. In particular $A$ preserves orientation if and only if $\det A > 0$.

**Definition 1.** *Let $\pi : E \to M$ with fiber $\mathbb{R}^n$ and connected base $M$ be a vector bundle. Let $\mathcal{U}, \phi_\alpha$ and $a_{\alpha\beta} : U_{\alpha\beta} \to GL(n, \mathbb{R})$ be as above. Then $E$ is* orientable *if and only if the cover $\mathcal{U}$ and the local trivia;izations $\pi_\alpha$ can be chosen so that for all $\alpha, \beta$, $a_{\alpha\beta} : U_{\alpha\beta} \to GL^+(n, \mathbb{R})$.*

*If $E$ is orientable, then an* orientation *of $E$ is an orientation of each fiber $E_p$ for all $p \in M$ that varies continuously with $p$. An orientation of $E$ is uniquely determined by the orientation of a single $E_{p_0}$, thus there are two possible choices for an orientaion of $E$.*

Note that the set of orientations on a vector space is a two-element set with the discrete topology, so continuous is the same as locally constant. Assume (as we always can by refinement) that the $U_\alpha$ are connected. Then each $E|_{U_\alpha} \cong U_\alpha \times \mathbb{R}^n$ has two orientations. Fix $\alpha$ and an orientation of $E|_{U_\alpha}$. If $U_\alpha \cap U_\beta \neq \emptyset$, since $a_{\alpha\beta}$ preserves orientaion, there is a unique compatible orientation in $U_\beta$, even if $U_{\alpha\beta}$ is disconnected. If $M$ is connected then for every $U_\gamma$ there is a "chain" $U_{\beta_1}, \dots, U_{\beta_k}$ joining $U_\alpha$ and $U_\gamma$ (all sucsesive intersections non-empty). Thus our definition of "orientability" guarantees the existence of an orientation.

### 2.2. **The Thom class.**

**Theorem 2.** *Let $\pi : E \to M$ be an oriented vector bundle with fiber $\mathbb{R}^n$ over a connected base $M$. Then*

(1) *$H^i(E, E \setminus 0) = 0$ for $i < n$*
(2) *$H^n(E, E \setminus 0, \mathbb{Z}) = \mathbb{Z}$. A generator $\mu$ is called a* Thom class *of $E$.*
(3) *For each $p \in M$, let $i_p : E_p \to E$ be the inclusion map. Then $i_p^* : H^n(E, E \setminus 0) \to H^n(E_p, E_p \setminus 0)$ is an isomorphism. In particular, the collection $\{i_p^* \mu : p \in M\}$ is an orientation of $E$.*
(4) Thom Isomorphism Theorem *: The map $H^k(M) \to H^{n+k}(E, E \setminus 0)$ taking $x \in H^k(M)$ to $\pi^* x \cup \mu$ is an isomorphism.*

*Proof.* These statements follow from the "Mayer-Vietoris principle". To write concise formulas, if $U \subset M$, write $E_U$ for $E|_U$ and $\mathcal{E}_U$ for the pair $(E_U, ], E_U \setminus 0)$. Similarly, if $p \in M$, write $\mathcal{E}_p$ for $(E_p, E_p \setminus 0)$. If $U, V \subset M$ are open, we have the

*Mayer-Vietoris sequence* on $M$:

(3)
$$H^{i-1}(U \cap V) \to H^i(U \cup V) \to H^i(U) \oplus H^i(V) \to H^i(U \cap V) \to H^{i+1}(U \cup V) \to \dots$$

and also on $\mathcal{E} = (E, E \setminus 0)$:

(4)
$$H^{i-1}(\mathcal{E}_{U \cap V}) \to H^i(\mathcal{E}_{U \cup V}) \to H^i(\mathcal{E}_U) \oplus H^i(\mathcal{E}_V) \to H^i(\mathcal{E}_{U \cap V}) \to H^{i+1}(\mathcal{E}_{U \cup V}) \to$$

Let us assume $M$ has a finite *good cover* $\mathcal{U} = \{U_1, \dots, U_k\}$ meaning that all non-empty intersections $U_{i_1} \cap \dots \cap U_{i_l}$, $l \geq 1$, are contractible. This is always possible if $M$ is a compact smooth manifold, by taking the convex open sets in a Riemannian metric. If $M = |K|$, the geometric realization of a finite simplicial complex $K$, take the cover by stellar neighborhoods of vertices.

Let $\mathcal{U}^1$ be the collection of these finite intersections. For $j = 2, 3, , \dots$ let $\mathcal{U}^j$ be the collection of open subsets of $M$ that are unions of at most $j$ elements of $\mathcal{U}^1$, and let $\mathcal{U}^\infty = \cup_{j=1}^\infty \mathcal{U}^j$. Note that every member of $\mathcal{U}^1$ is contractible and $M \in \mathcal{U}^k \subset \mathcal{U}^\infty$. The strategy is to use the Mayer - Vietoris sequences to prove, by induction on $j$, any statement on $M$ that can be formulated for arbitrary open subsets $U \subset M$ and is true for contractible $U$. For the inductive step, each element of $\mathcal{U}^{j+1}$ is of the form $U \cup V$ for some $U \in \mathcal{U}^j$ and some $V \in \mathcal{U}^1$. But then $U \cap V \in \mathcal{U}^j$, and this is the setting for Mayer-Vietoris.

For the first statement, its clearly true for $U \in \mathcal{U}^1$. Suppose it is true for all $U \in \mathcal{U}^j$, and let $V \in \mathcal{U}^1$. Then the sequence (4) for $i < n$ gives

$$H^{i-1}(\mathcal{E}_{U \cap V}) \to H^i(\mathcal{E}_{U \cup V}) \to H^i(\mathcal{E}_U) \oplus H^i(\mathcal{E}_V)$$

By the induction hypothesis the first and third term vanish, so must the middle term $H^i(\mathcal{E}_{U \cup V})$, and the induction is complete.

To prove the second and third statements by this method, we need to be prove the following statement for an arbitrary $U \in \mathcal{U}^\infty$:

*Let $\{\eta_p : p \in M\}$ be an orientation of $E$ as in Definition 1: $\eta_p \in H^n(\mathcal{E}_p, \mathbb{Z})$ is a generator, depending continuously on $p$. Then for every open set $U \in \mathcal{U}^\infty$ there exists a unique element $\mu_U \in H^n(\mathcal{E}_U, \mathbb{Z})$ with the property that for all $p \in U$, $i_p^*(\mu_U) = \eta_p$.*

Recall that continuous dependence means locally constant: for each $p \in U$ and contractible neighborhood $W$, $p \in W \subset U$, for any trivialization $\mathcal{E}_W \cong W \times \mathcal{E}_p$, then $\eta_q = i_q^*(1 \otimes \eta_p)$ (where $1 \otimes \eta_p \in H^n(W \times \mathcal{E}_p)$ is the class corresponding to $\eta_p$ under the Künneth isomorphism.)

To prove this statement, first it is true for $U$ contractible, since then $\mathcal{E}_U \cong U \times \mathcal{E}_{p_0}$ for any fixed $p_0 \in U$, and $\mu_U$ is the class corresponding to $1 \otimes \eta_{p_0}$ under this isomorphism. In partcular, the statement is true for $U \in \mathcal{U}^1$.

Suppose that it is true for all $U \in \mathcal{U}^j$ and suppose $V \in \mathcal{U}^1$. Then the sequence (4) for $i = n$ gives

$$0 \to H^n(\mathcal{E}_{U \cup V}) \to H^n(\mathcal{E}_U) \oplus H^n(\mathcal{E}_V) \xrightarrow{i^*_{U \cap V, V} - i^*_{U \cap V, V}} H^n(\mathcal{E}_{U \cap V}), \tag{5}$$

where the vanishing of the first term is a consequence of the proof of the first statement: $H^{n-1}(U \cap V) = 0$ if $U \cap V \in \mathcal{U}^\infty$.

Consider the continuous family $\{\eta_p \in H^n(\mathcal{E}_p, \mathbb{Z}) : p \in U \cup V\}$ of orientations of the $\mathcal{E}_p, p \in M$, restricted to $U \cup V$. Restrict these to $U$ and $V$. By induction there is a unique $\mu_U \in H^n(\mathcal{E}_U, \mathbb{Z})$ such that $i_p^*(\mu_U) = \eta_p$ for all $p \in U$, and a similar $\mu_V \in H^n(\mathcal{E}_V, \mathbb{Z})$. These restrict to classes $i^*_{U\cap V,U}(\mu_U) \in H^n(\mathcal{E}_{U\cap V})$ and $i^*_{U\cap V,V}(\mu_V) \in H^n(\mathcal{E}_{U\cap V})$ with the same property as $\mu_{U\cap V}$. By the induction hypothesis these classes are all equal, in particular, $i^*_{U\cap V,U}(\mu_U) - i^*_{U\cap V,V}(\mu_V) = 0$. Thus the sequence (5) implies that there is a unique class $\mu_{U\cup V}$ that restricts to $\mu_U$ and $\mu_V$.

Note that the uniqueness of $\mu_U$ has the following important consequence:

$$\text{If } U, V \in \mathcal{U}^\infty \text{ and } U \subset V, \text{ then } i^*_{U,V}(\mu_V) = \mu_U. \tag{6}$$

Next, we prove the Thom isomorphism, from which all the remaining statements will follow. For $U \in \mathcal{U}^\infty$, consider the maps $H^k(U) \overset{\pi^*\cup\mu}{\longrightarrow} H^{k+n}(\mathcal{E}_u)$. The naturality (6) implies that for any $U, V \in \mathcal{U}^\infty$ we get a map of the Mayer-Vietoris sequence (3) on $M$ to the correspoding one (4) on $\mathcal{E}$. The maps are isomorphisms if $U \in \mathcal{U}^1$, and the induction step follows from the "five lemma":isomorphim for $U, V$ and $U \cap V$ implies isomorphism for $U \cup V$. This proves that $H^k(M, \mathbb{Z}) \overset{\pi^*\cup\mu}{\longrightarrow} H^{k+n}(M, \mathbb{Z})$ is an isomorphism. In particular, since $H^0(M, \mathbb{Z}) = \mathbb{Z}$, we get $H^n(\mathbb{E}, \mathbb{Z}) = \mathbb{Z}$ and therefore $i_p^* : H^n(\mathcal{E}) \to H^n(\mathcal{E}_p)$ is isomorphic for all $p \in M$. $\square$

2.2.1. *Integration over the fiber.* The construction of the Thom class in the last proof may seem somewhat mysterious. As often happens in topology, there are several models for constructing invariants, and a specific construction may be easier to understand in some particular model. For example, cohomology may be represented as singular cohomology, or de Rham cohomolgy, or Chech,... Some specific characteristic classes may be more natural in one of these theories than in any of the others;

For the Thom isomorphism, there is a very natural construction for its inverse $\pi_* : H^{n+k}(E, E \setminus 0) \to H^k(M)$ by using differential forms and *integration over the fiber*. This is explained very well in [1], so we give a brief description.

First of all, relative cochains on $(E, E \setminus 0)$ cannot be represented by differential forms, since a non-zero form cannot vanish on a dense open set. But the same cohomology (with $\mathbb{R}$-coefficients) can be achived by the complex of forms with *fiber-compact supports*. Let $\pi : E \to M$ be a smooth vector bundle with local trivializations $\phi_\alpha : E_{U_\alpha} \to U_\alpha \times \mathbb{R}^n$ and transition functions $a_{\alpha\beta} : U_{\alpha\beta} \to GL^+(n, \mathbb{R})$ as above. Let $A^*(E)$ denote the deRham complex of differential forms on the space $E$ and let $A^*_{fc}(E)$ denote the set of all $\eta \in A^*(E)$ with the property that for all $\alpha$ there exists a compact set $K_\alpha \subset \mathbb{R}^n$ so that the support of $\eta|_{U_\alpha\times\mathbb{R}^n}$ is contained in $U_\alpha \times K_\alpha$.

This definition is independent of choices and closed under exterior differentiation. One could compare the $K_\alpha$ to be balls in some Riemannian metric and write (for a compact base, or a base with a finite cover of trivializing $U_\alpha$'s) $A^*_{fc}(E) = \cup_n A^*(B_n(E))$, the associated bundle of balls of radius $n$ in $E$, and get deRham maps

$$A^*_{fc}(E) \xrightarrow{\int} \bigcup_n C^*(E, E \setminus B_n(E))$$

and use the fact that the singular cohomoogy groups obtained from the right-hand side are all isomorphic to $H^*(E, E \setminus 0)$.

Now $A^m_{fc}(U_\alpha \times \mathbb{R}^n) = \oplus_{i+j=m} A^{i,j}_{fc}(U_\alpha \times \mathbb{R}^n)$, where, in multi-index notation,

$$A^{i,j}_{fc}(U_\alpha \times \mathbb{R}^n) = \{ \sum_{|I|=i, |j|=j} \phi_{I,J}(x,y) dx^I dy^J : \phi_{I,J} \in A^0_{fc}(U_\alpha \times \mathbb{R}^n)\}$$

If $m = k + n \geq n$, then

$$A^{k,n}_{fc} = \{ \sum_{|I|=k} \phi_I(x,y) dx^I dy^1 \dots dy^n : \phi_I(x,y) \in A^0_{fc}(U_\alpha \times \mathbb{R}^n)\}$$

and $\int_F : A^{k,n}_{fc}(U_\alpha \times \mathbb{R}^n) \to A^k(U_\alpha)$ is defined by

$$\int_F (\phi_I(x,y) dx^I dy^1 \dots dy^n) = (\int_{y \in \mathbb{R}^n} \phi_I(x,y) dy^1 \dots dy^n) dx^I$$

and is extended to $A^*_{fc}$ by setting it to be zero on $A^{i,j}_{fc}$ whenever $j \neq n$.

One needs to verify that this is well defined and gives a chain map (commuting with $d$) thus a well defined map in cohomology. This is $\pi_* : H^{k+n}_{fc}(E) \to H^k(M)$. It is easily proved to be an isomorphism by using the Mayer-Vietoris arguments as in the proof of Theorem 2. The Thom class $\mu$ is characterized by $\pi_*(\mu) = 1$.

2.2.2. *The Gysin Sequence.*

**Theorem 3.** Gysin Sequence*: Let $\pi : E \to M$ be an oriented vector bundle with fiber $\mathbb{R}^n$. Then there is an exact sequence*

$$\dots \to H^0(M) \xrightarrow{e(E)} H^n(M) \xrightarrow{\pi^*} H^n(E \setminus 0) \xrightarrow{\pi_*} H^1(M) \to \dots$$

*where $e(E) \in H^n(M)$ is defined to be the image of $\mu$ under the natural maps $H^n(E, E \setminus 0) \to H^n(E) \cong H^n(M)$.*

*Proof.* The Gysin sequence follows from the Thom isomorphism applied to the exact sequence of the pair $(E, E \setminus 0)$:

$$\begin{array}{ccccccc}
H^n(E, E\setminus 0) & \xrightarrow{j} & H^n(E) & \xrightarrow{i*} & H^n(E\setminus 0) & \xrightarrow{\delta_*} & H^{n+1}(E, E\setminus 0) \\
\pi^* \cup \mu \uparrow & & \pi^* \uparrow & & id \uparrow & & \pi^* \cup \mu \uparrow \\
H^0(M) & \xrightarrow{e(E)} & H^n(M) & \xrightarrow{\pi^*} & H^n(E\setminus 0) & \xrightarrow{\pi_*} & H^1(M)
\end{array}$$

□

All maps in this diagram have geometric significance and should be examined closely, In particular, we display more carefully the definition of Euler class given in the statement of the theorem:

**Definition 2.** *The* Euler class $e(E) \in H^n(M)$ *is* $((\pi^*)^{-1} \circ j)(\mu)$.

If $E \to M$ is an oriented vector bundle with fiber $\mathbb{R}^n$ over an oriented manifold of dimension $n$, and $s : M \to E$ is a section with an isolated zero at $p \in M$ then the *index of $s$ at $p$* is defined as follows: choose a local trivialization of $E|_U \cong U \times \mathbb{R}^n$ over a neighborhood $U$ of $p$, let $f : (U, U \setminus p) \to (\mathbb{R}^n, \mathbb{R}^n \setminus 0)$ correspond to $s$: $s(x) = (x, f(x))$. Then the local index is defined as $\iota_p(s) = \deg(f)$, where the chosen orientations on $M$ and $E$ are used in defining the degree. Thus $\iota_p(s)$ depends on two choices of orientation,

**Proposition 5.** *Let $\pi : E \to M$ be an oriented vector bundle with fiber $\mathbb{R}^n$ over a closed, oriented manifold $M$of dimension $n$. Let $s : M \to E$ be a continuous section of $E$ with isolated zeros. Then*

$$e(E)[M] = \sum_{s(p)=0} \iota(p).$$

*Proof.* Consider the diagram

$$\begin{array}{ccccc}
\bigoplus_{s(p)=0} H^n(E|U_p, E|U_p \setminus 0) & \xleftarrow{i^*} & H^n(E, E \setminus 0) & \xrightarrow{j} & H^n(E) \\
s^* \downarrow & & s^* \downarrow & & s^* \downarrow \\
\bigoplus_{s(p)=0} H^n(U_p, U_p \setminus p) & \xleftarrow{i^*} & H^n(M, M \setminus \{s = 0\}) & \xrightarrow{j} & H^n(M).
\end{array} \tag{7}$$

Start with the Thom class $\mu \in H^n(E, E \setminus 0)$. Going right then down we get $s^* j(\mu) = e(E)$ since $s^* = (\pi^*)^{-1}$. Going down, then right we get $js^*(\mu) \in H^n(M)$ which is a class supported on the zero set. This means that it is the image of a class in $H^n(M, M \setminus 0)$, namely $s^*(\mu)$. Since $H^n(M, M \setminus 0) \xleftarrow{i^*} \oplus H^n(U_p, U_p \setminus p)$ is an isomorphism, we see that $i^* s^*(\mu)$ is the "vector" $i^* s^*(\mu)$ with components the local indices of $s$. The map $j$ is gives the sum of the components. More precisely, $js^*(\mu)([M]) = \sum_{s(p)=0} \iota_p(s)$ while $s^* j(\mu)([M]) = e(E)([M])$

. ☐

Proposition 5 justifies the first step in Hopf's proof of Theorem 1 in case that $M$ is orientable. The non-orientable case reduces to this by passing to the orientation double cover: both sides of the equality are defined for all manifolds, independent of orientation, and both sides are multiplicative under coverings. Thus the proof ot the Poincaré-Hopf Theorem (Theorem 1) is complete.

2.3. **The Poincaré dual class of a submanifold.** If $M^{n+k}$ is a closed oriented manifold and $X^k \subset M$ is a closed oriented submanifold (superscripts denote dimension), then $X$ defines a homology class $[X] \in H_k(M, \mathbb{Z})$ which has a Poincaré dual class $\widehat{[X]} \in H^n(M, \mathbb{Z})$. Working modulo torsion, $\widehat{[X]}$ is characterized by the identity

$$\alpha([X]) = (\alpha \cup \widehat{[X]})([M]) \text{ for all } \alpha \in H^k(M). \tag{8}$$

We describe an explicit construction of $\widehat{[X]}$. Let $N$ be a tubular neighborhood of $X$ in $M$, diffeomorphic to the total space of the normal bundle of $X$ in $M$. Thus $N$ is a neighborhood of $X$ in $N$ so that there is a projection $\pi : N \to X$ which is diffeomorphic to a vector bundle, with $X \subset N$ as the zero-section. Then we have an inclusion $i : (N, N\setminus 0) \to (M, M\setminus X)$ which by excision gives an isomoprhism $i^* : H^*(M, M \setminus X) \xrightarrow{\cong} H^*(N, N \setminus 0)$. Let $\tilde{j} : H^*(N, N \setminus 0) \to H^*(M)$ be the composition

$$H * (N, N \setminus 0) \xrightarrow{(i^*)^{-1}} H^*(M, M \setminus X) \xrightarrow{j} H^*(M), \tag{9}$$

where $j : H^*(M, M \setminus X) \to H^*(M)$ is the relative to absolute map in the exact sequence of the pair $(M, M \setminus X)$

**Proposition 6.** $\widehat{[X]} = \tilde{j}(\mu)$, *where $\mu$ is the Thom class of the normal bundle $N$ of $X$ in $M$.*

*Proof.* Let $\alpha \in H^k(M)$. Then $\alpha([X]) = i_X^*\alpha([X])$, where $i_X : X \to M$ is the inclusion. The Thom isomorphism $H^k(X) \xrightarrow{\pi^*\cup\mu} H^{n+k}(N, N \setminus 0)$ between two groups, each isomorphic to $\mathbb{Z}$ by evaluation on the respective fundamental classes, gives us $\imath_X^*\alpha([X]) = ((\pi^* i_X^*\alpha) \cup \mu)([N, N \setminus 0])$ (clear at least up to sign, which is enough for a first look). Since the projection $\pi : N \to X$ is a homotopy equivalence, $\pi^* i_X^* \alpha = i_N^*\alpha$ . Now $\alpha \cup \tilde{j}(\mu) = i_N^*\alpha \cup \mu$, evaluating on respective fundamental classes gives (8). $\square$

Geometrically, since (8) asserts the equality of two homomorphisms $H^k(M) \to \mathbb{Z}$ and the first is supported on $X$, so must the second. This means that the cohomology class $\widehat{[X]}$ should also be supported on $X$, meaning it should be in the image $H^n(M, M \setminus X) \to H^n(M)$. Since $H^n(M, M \setminus X) \cong H^n(N, N \setminus X) = \mathbb{Z}\mu$, must have that $\widehat{[X]} = \tilde{j}(m\mu)$ for some $m \in \mathbb{Z}$. Proposition 6 asserts $m = 1$.

2.3.1. *Transversality.* Let $f : M_1 \to M_2$ be a smooth map of smooth manifolds, and let $X_2 \subset M_2$ be a smooth submanifold. We say that $f$ is transversal to $X_2$ if and only if, for each $p \in f^{-1}(X_2)$ the composition

$$T_pM_1 \xrightarrow{d_pf} T_{f(p)}M_2 \xrightarrow{P} T_{f(p)}M_2/T_{f(p)}X_2 = N_{f(p)}X_2 \tag{10}$$

is surjective, where $NX_2$ denotes the *normal bundle* of $X_2$ in $M_2$ and $P : TM_2|_{X_2} \to NX_2$ denotes the projection onto the quotient ($NX_2$ can be viewed either as a quotient of $TM_2|_{X_2}$, as indicated in (10), or a sub-bundle of $TM_2|_{X_2}$ complementary to $TX_2$. The second description is compatible with the embedding of $NX_2$ in $M_2$ as a tubular neighborhood).

Standard arguments based on the implicit function theorem give

**Theorem 4.** *Suppose $f : M_1 \to M_2$ is transverse to $X_2 \subset M_2$ as above. Then*

(1) *$X_1 = f^{-1}(X_2)$ is a submanifold of $M_1$ with tangent bundle $TX_1 =$* $\ker(Pdf)$.
(2) *$df$ induces an isomorphism $df : NX_1 \to f^*NX_2$. In particular, $NX_1 \cong f^*NX_2$.*

(3) *The Thom classes of* $NX_1, NX_2$ *satisfy* $\mu_{NX_1} = f^*(\mu_{NX_2})$.
(4) *The Poincaré dual classes satisfy* $\widehat{[X_1]} = f^*(\widehat{[X_2]}$

*Proof.* The first assertion follows from the implicit function theorem, the second mostly by definition, the third by the defining properties of the Thom class, and the fourth from Proposition 6. □

**Remark 1.** *Observe that* $X_1$ *need not be connected, even if* $X_2$ *is connected. For disconnected* $X_1$*, the Thom class of* $NX_1$ *means the sum of the Thom classes of its connected components.*

We can now state and prove the generalization of Proposition 5 to the case when the base has possibly higher dimension than the fiber:

**Proposition 7.** *Let* $\pi : E \to M$ *be an oriented vector bundle with fiber* $\mathbb{R}^n$ *over the closed oriented manifold* $M$ *of dimension* $m$*, where* $m \geq n$*, let* $s : M \to E$ *be a section of* $E$ *which is* transverse *to the zero section, and let* $Z_s = \{x \in M : s(x) = 0\} \subset M$ *be the zero set of* $s$*. Then* $Z_s$ *is a submanifold of* $M$ *which is Poincaré dual to the Euler class* $e(E)$*:*

$$e(E) = \widehat{[Z_s]} \in H^n(M).$$

*Proof.* Consider the commutative square in the right of the diagram (7) used in the proof of Proposition 5. It also makes sense in the context $m > n$ (while the rest of (7) does not).

By the first and parts of Theorem 4, $Z_s$ is a submanifold of $M$ with normal bundle $s^*E$ and the Thom class of this bundle is $s^*\mu$. By Proposition 6, $js^*\mu = \widehat{[Z_s]} \in H^n(M)$, while $s^*j(\mu) = e(E)$, almost by definition (Definition 2) since $s^* = (\pi^*)^{-1}$. □

This proposition illustrates a basic principle: *Characteristic classes are Poincaré dual to singularities.*

Finally, the naturality of the Thom class gives us that if $f : M_1 \to M_2$ and $E \to M_2$ is an oriented vector bundle with fiber $\mathbb{R}^n$, and $f^*E \to M_1$ is the induced bundle, then

$$e(f^*E) = f^*e(E) \text{ in } H^n(M_1, \mathbb{Z}). \tag{11}$$

In other words, the Euler class is *functorial*, illustrating the basic principle: *characteristic classes are functiorial*.

## 3. Obstruction Theory

The discussion of the Euler class fits into the more general context of *obstruction theory*. Suppose we have

(1) A locally trivial fibration $p : E \to B$ with fiber $F$.
(2) The base is triangulated: $B = |K|$ for some simplicial complex $K$.
(3) The fiber is $(r-1)$-connected: $\pi_i(F) = 0$ for $i < r$, and $\pi_r(F) \neq 0$.
(4) If $r = 1$, $\pi_1(F)$ is abelian.

The goal of obstruction theory is to determine when $p : E \to B$ has a section. The strategy is to do it skeleton by skeleton, and see what happens. Before we start, for each simplex $\sigma < K$ choose, once and for all, a bundle isomorphism

$$h_\sigma = (p, h_\sigma^F) : E|_{|\sigma|} \to |\sigma| \times F, \tag{12}$$

where we explicitly write the forms of $h_\sigma$ followed by projection to each factor. Namely, followed by projection to $|\sigma|$ must get $p$, followed by projection to $F$ get a map we call $h_\sigma^F$.

This isomorphisms need not be compatible, and would not be unless $p : E \to B$ is the trivial fibration $p_B : B \times F \to B$. The important point is that they be fixed in advance.

Once all the $h_\sigma$ are fixed, we can proceed in steps as follows:

(1) There is always a section $s^0$ defined over $K^0$: to each vertex (0-simplex) $v$, pick a point $s^0(v) \in E_v$.

(2) Try to extend $s^0$ to a section $s^1$ over $K^1$:

  (a) If $\pi_0(F) = 0$, that is, if $F$ is connected, then, for each one-simplex $\sigma =< v_0, v_1 >< K$, there is a path $\gamma$ in $F$ from $h_\sigma^F(s^0(v_0))$ to $h_\sigma^F(s^0(v_1))$. Then

$$((1-t)v_0 + tv_1, \gamma(t))$$

  defines a section of $|\sigma| \times F$ that goes under $h_\sigma^{-1}$ to a section of $p$ over $|\sigma|$ that extends $s^0$ over $\partial\sigma$. Doing this over every one-simplex we get a section $s^1$ over $K^1$ that extends $s^0$.

  (b) If $\pi_0(F) \neq 0$, that is, if $F$ is disconnected, then we have to work harder. The next step would be to define a one-cochain $\mathfrak{o}(s^0) \in C^1(K, \pi_0(F))$ by

$$\mathfrak{o}(s^0)(\sigma) = [h_\sigma^F(s^0(v_1)) - h_\sigma^F(s^0(v_0))] \in \pi_0(F) \tag{13}$$

  where the bracket $[.]$ denotes the homotopy class. There is an obvious problem with this formula: $\pi_0(F)$ need not be a group. But the statements $\mathfrak{o}(s^0)(\sigma) = 0$, $\mathfrak{o}(s^0)(\sigma) \neq 0$ have a clear meaning: the two points lie in the same connected component of $F$, or in different components, respectively. For the moment, let us interpret (13) in this way. It is then clear that *$s^0$ extends over $K^1$ if and only if $\mathfrak{o}(s^0) = 0$.* This is essentially an obvious re-statement of what it means to extend $s^0$ over $K^1$, and not very useful. The useful statement would be: $\mathfrak{o}(s^0)$ *is a cocycle, and it is a coboundary if and only if $s^0$ can be replaced by a section $t^0$ over $K^0$ that extends over $K^1$.*

(3) This is indeed the case. But given the exceptional nature of $\pi_0$ and $\pi_1$, let us assume that $F$ is connected and simply connected, so that the discussion starts with $\pi_2(F)$ which is an abelian group. Then arguing as in (a) above we get that $s^0$ extends to a section $s^2$ over $K^2$. Then to extend over $K^3$, look at a 3-simplex $\sigma < K$.

(a) If $\pi_2(F) = 0$, then $h_\sigma^F(s^2|_{|\partial\sigma|})$ extends to a map $\phi : |\sigma| \to F$, and $(x, f(x)),\ x \in |\sigma|$ defines a section of $|\sigma| \times F$ that goes under $h_\sigma^{-1}$ to a section of $p$ over $|\sigma|$ that extends $s^2$. This would be $s^3$ on $|\sigma|$.

(b) If $\pi_2(F) \neq 0$, then, in analogy with (13) define a 3-cochain $\mathfrak{o}(s^2) \in C^3(K, \pi_2(F))$ by

$$\mathfrak{o}(s^2)(\sigma) = [(h_\sigma^F \circ s^2)|_{|\partial\sigma|}] \in \pi_2(F). \tag{14}$$

It is clear that $s^2$ extends over the 3-skeleton if and only if $\mathfrak{o}(s^2) = 0$. We will prove that $\mathfrak{o}(s^2)$ is a cocycle, and is a coboundary if and only if we can replace $s^2$ by a section $t^2$ that agrees with $s^2$ on the one-skeleton $K^1$ and extends over $K^3$.

The general procedure is now clear. But we need to be careful about details. First we have to make the coefficients precise: $\pi_r(F)$ is a *local coefficient system* on $K$. For our purposes a local coefficient system on $K$ means an assignment of an abelian group $A_v$ to each vertex $v$ of $K$, and to each edge $< v_0, v_1 >$ if $K$ an isomorphism $< v_0, v_1 >^\#$: $A_{v_1} \xrightarrow{\cong} A_{v_0}$ so that

$$< v_0, v_1 >^\# < v_1, v_2 >^\# = < v_0, v_2 >^\# \quad \text{if} \quad < v_0, v_1, v_2 > < K. \tag{15}$$

All the $A_v$ are isomorphic, say to a fixed abelian group $A$, but not "equal". In fact (15 ) is equivalent to an action of $\pi_1(B)$ on $A$ by a group $G$ of automorphisms of $A$, and the collection of $A_v$ determoine a bundle over $B$ with fiber $A$ and stucture group $G$. This bundle may be the product bundle $B \times A$, or may be non-trivial.

This structure is exactly what it's needed to define cohomology with coefficients in $A$: First our simplices $\sigma = < x_0, \dots, x_k >$ will always be *ordered*, and $x_0$ will be called the *leading vertex* of $\sigma$. For each $\sigma$ we define $A_\sigma = A_{x_0}$. In this way we assign a group $A_\sigma$ to each ordered simplex $\sigma$. Then we define the space of *$k$-cochains with coefficients in $\underline{A}$* by

$$C^k(K, \underline{A}) = \bigoplus_{\sigma < K, \dim(\sigma) = k} A_\sigma \tag{16}$$

with coboundary defined as follows:

$$\delta c(\sigma) = < x_0, x_1 >^\# c(\sigma^{(0)}) + \sum_{j=1}^{k+1} (-1)^j c(\sigma^{(j)}) \tag{17}$$

where $\sigma = < x_0, \dots, x_{k+1} >$ is an arbitrary $k+1$-simplex and for $j = 0, \dots k+1$, $\sigma^{(j)} = < x_0, \dots, \hat{x_j}, \dots, x_{k+1} >$ is the face of $\sigma$ opposite to the $j^{th}$ vertex $x_j$. Observe that for $j > 0$, $c(\sigma^{(j)}) \in A_{x_0}$, while $c(\sigma^{(0)}) \in A_{x_1}$, thus $< x_0, x_1 >^\#$ $c(\sigma^{(0)}) \in A_{x_0}$, so this formula makes sense.

The equation $\delta^2 = 0$ is a consequence of (15) and the cohomology of this complex is denoted by $H^*(K, \underline{A})$.

Now given our fibration $p : E \to B = |K|$ with $(r-1)$-connected fiber $F$ we can define the local coefficient system $\underline{\pi_r(F)}$ by

**Definition 3.** *Let $\underline{\pi_r(F)}$ be the local system on $K$ defined as follows: For each vertex $v$ of $K$ define $\underline{\pi_r(F)}_v = \pi_r(E_v)$. For each edge $< v_0, v_1 > < K$, let $< v_0, v_1 >^{\#}$: $\pi_r(E_{v_1}) \to \pi_r(E_{v_0})$ be the homomophism induced by "parallel translation" $E_{v_1} \to E_{v_0}$ along the path $< v_0, v_1 >$.*

Consequently, for each ordered simplex $\sigma =< x_0, \dots, x_k >, \pi_r(F)_\sigma = \pi_r(E_{x_0})$. Note that we have not specified a base-point for $\pi_r(F)$ or for $\pi_1(E_{x_0})$. The reason is that for all $x, y \in F$ and for all paths $\gamma_1, \gamma_2$ in $F$ from $x$ to $y$ the induced isomorphisms $\gamma_i^{\#} : \pi_r(F, y) \to \pi_r(F, x)$ coincide: $\gamma_1^{\#} = \gamma_2^{\#}$, in other words, $\pi_r(F, x)$ is determined, as $x$ varies, up to *unique isomorphism*. The reason is very simple: if $r \geq 2$ then $\pi_1(F)$ is trivial and hence any two paths $\gamma_1, \gamma_2$ from $x$ to $y$ are homotopic relative to the endpoints. If $r = 1$ we have assumed that $\pi_1(F)$ is abelian, so it has no non-trivial inner automorphisms. Since $\gamma_1^{\#} = \gamma_2^{\#}$ times an inner automorphism, we get that for all $\gamma_1, \gamma_2$, $\gamma_1^{\#} = \gamma_2^{\#}$.

Now the situation with respect to the groups $\pi_r(E_{x_0})$ is quite different. Given $x_0, x_1 \in B$ the and paths $\gamma_1, \gamma_2$ from $x_0$ to $x_1$ the isomorphisms $\gamma_1^{\#}$ and $\gamma_2^{\#}$ could be different if $\pi_1(B)$ operates non-trivially on $\pi_r(F)$.

**Theorem 5.** *Let $p : E \to B$, $F$, $r$, $K$, $\underline{\pi_r(F)}$, $h_\sigma$ be as above.*

(1) *There exists a section $s : |K^r| \to E|_{|K^r|}$. If $s, s'$ are any two sections on $|K^r|$, their restrictions to $|K^{r-1}|$ are homotopic.*
(2) *Given a section $s$ over $|K^r|$, let $\mathfrak{o}(s) \in C^{r+1}(K, \underline{\pi_r(F)})$ be defined as in (14): for each $(r+1)$-simplex $\sigma =< x_0 \dots, x_{r+1} >$,*

$$\mathfrak{o}(s)(\sigma) = [\phi_\sigma \circ s|_{|\partial\sigma|}] \in \pi_r(E_{x_0}) \tag{18}$$

*where $\phi_\sigma : E|_\sigma \to E_{x_0}$ is the projection defined by the product structure $h_\sigma$ of (12), namely $\phi_\sigma(z) = h_\sigma^{-1}((x_0, h_\sigma^F(z)))$.*

Then*:*

(a) *$s$ extends over $K^{r+1}$ if and only if $\mathfrak{o}(s) = 0$.*
(b) *$\mathfrak{o}(s)$ is a cocyle.*
(c) *If $s'$ is a section over $K^r$ that agrees with $s$ on $K^{r-1}$, then $\mathfrak{o}(s')$ is cohomologous to $\mathfrak{o}(s)$.*
(d) *$\mathfrak{o}(s)$ is a coboundary if and only if $s$ can be redefined on $K^r$, leaving it unchanged on $K^{r-1}$, to a section $s'$ that extends to $K^{r+1}$.*

*Proof.* We have already explained that $\pi_i(F) = 0$ for $i < r$ implies the existence of sections over the $r$-skeleton $K^r$. The same inductive argument applied to the cell complex $K \times I$, (where $I$ is the unit interval and the cells are $\sigma \times 0, \sigma \times 1, \sigma \times I$ for simplices $\sigma < K$) gives homotopies between any two sections as long as the boundaries of the cells $\sigma \times I$ have dimension $< r$, that is, as long as $\dim(\sigma) < r$. Thus part (1) is clear.

For the second part, we have also explained (2a), which is clear from the definition of $\mathfrak{o}(s)$: $s$ extends to $K^{r+1}$ if and only if $\mathfrak{o}(s) = 0$. To prove (2b), that $\mathfrak{o}(s)$ is a cocycle, fix an $(r+2)$-simplex $\Delta_{r+2} =< x_0, \dots, x_{r+2} >$. To compute $\delta\mathfrak{o}(s)(< x_0, \dots, x_{r+2} >)$ it suffices to restrict $E$ and its associated constructions to the subcomplex $\Delta_{r+2} < K$.

Since $|\Delta_{r+2}|$ is contractible, there exists a bundle isomorphism $E|_{|\Delta_{r+2}|} \xrightarrow{\cong} |\Delta| \times E_{x_0}$ which resuts in *unique* isomorphisms $\pi_r(E_x) \xrightarrow{\cong} \pi_r(E|_{|\Delta_{r+2}|}) \xleftarrow{\cong} \pi_r(E_{x_0})$ for each $x \in |\Delta_{r+2}|$, and which agree with the parallel translation isomorphisms. Keeping this in mind, if we look at the $r+3$ terms of $\delta\mathfrak{o}(< x_0, \ldots, x_{r+2} >)$ given by (17), together with the definition (18) we see that we can interpret the formula as follows:

Let $\Delta_{r+1}$ denote an abstract $(r+1)$-simplex, and let $\iota_0, \ldots, \iota_{r+2}$ be the embeddings $\iota_j : \Delta_{r+1} \to \Delta_{r+2}$ as the face opposite the $j^{th}$ vertex. These result in $r+3$ embeddings $\iota_j : \Delta^r_{r+1} \to \Delta^r_{r+2}$ where the $r$-skeleton $\Delta^r_{r+1} = \partial\Delta_{r+1}$ has geometric realization homeomorphic to the $r$-sphere $S^r$.

Our section $s$ over $K^r$ restricts to a section $s$ over $\Delta^r_{r+2}$. Under the product structure $s(X)$ corresponds to $(x, f(x))$ for a map $f : |\Delta^r_{r+2}| \to E_{x_0}$. Namely, in the notation of (18), $f = \phi_{\Delta_{r+2}}$. We get $r+3$ maps $f \circ \iota_j : |\Delta^r_{r+1}| \to E_{x_0}$. Let $\eta \in \pi_r(|\Delta^r_{r+1}|)$ be a generator. We see that $\delta\mathfrak{o}(s)(\Delta_{r+2})$ is the element $< x_0, x_1 >^\# (f \circ \iota_0(\eta)) + \sum(-1)^j(f \circ \iota_j)(\eta)$ of $\pi_r(E_{x_0})$. To see that this is zero, apply the Hurewicz homomorphism:

$$\begin{array}{ccccc}
\pi_r(|\Delta^r_{r+1}|) & \xrightarrow{(\hat\iota_0)_* - (\iota_1)_* +, \ldots} & \pi_r(|\Delta^r_{r+2}|) & \xrightarrow{f_*} & \pi_r(E_{x_0}) \\
H\big\downarrow & & H\big\downarrow & & H\big\downarrow \\
H_r(|\Delta^r_{r+1}|) & \xrightarrow{(\iota_0)_* - (\iota_1)_* + \ldots} & H_r(|\Delta^r_{r+2}|) & \xrightarrow{f_*} & H_r(E_{x_0})
\end{array}$$

(where $(\hat\iota_0)_* =< x_o, x_1 >^\# (\iota_0)_*$ is needed in the top row, not in the bottom.). The vertical arrows are the Hurewicz homomorphisms $H$. Since all the spaces involved are $(r-1)$-connected, all the vertical arrows are isomorphisms. We will only need that the last one is an isomorphhism.

Finally, it is easy to see that the map $\sum_{j=0}^{r+2}(-1)^j(\iota_j)_* : H_r(\Delta^r_{r+1}) \to H_r(\Delta^r_{r+2})$ is the zero map. This is the familiar argument for $\partial^2 = 0$ illustrated in Figure 4. Thus $H(\delta\mathfrak{o}(s)) = 0$, therefore, $\delta\mathfrak{o}(s) = 0$ and $\mathfrak{o}(s)$ is a cocycle. This concludes the proof of (2b).

For (2c), suppose $s$ and $s'$ are two sections on $|K^r|$ that agree on $|K^{r-1}|$: Define an $r$-cochain $\mathfrak{d}(s, s') \in C^r(K, \underline{\pi_r(E_{x_0})})$, called the *difference cochain* of $s, s'$, as follows. Given any $r$-simplex $\sigma =< x_0, \ldots, x_r >$, observe that $s|_{|\partial\sigma|} = s'|_{|\partial\sigma|}$. Form a standard sphere $S^r$ by taking the disjoint union of two copies of $|\sigma|$ identified by the identity map of $|\partial\sigma|$ There is a well-defined map $d(s, s') : S^r \to E|_{|\sigma|}$ to be $s'$ on the upper hemisphere and $s$ on the lower hemisphere, and following by projection to $E_{x_0}$, a map $S^r \to E_{x_0}$. In other words, $d(s, s')(\sigma)$ is the composition

$$S^r = \sigma \sqcup_{\partial\sigma} \sigma \xrightarrow{s' \sqcup s} E|_\sigma \to E_{x_0}. \tag{19}$$

We define $\mathfrak{d}(s, s')(\sigma) \in \pi_r(E_{x_0})$ to be the homotopy class of this map. For later use, record this as a definition:

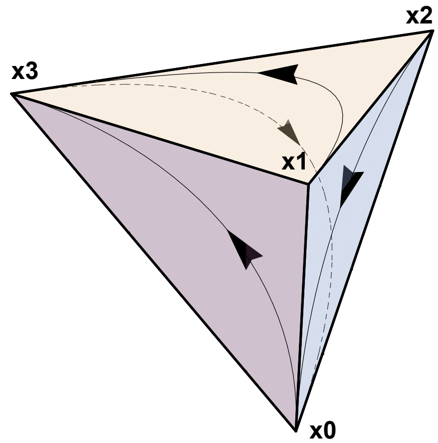


FIGURE 4. Picture for $r = 1$ of $\delta\mathfrak{o}(s) = 0$

**Definition 4.** *Let $s, s'$ be two sections of $E|_{|K^r|}$ that agree on $|K^{r-1}|$. The cochain $\mathfrak{d}(s, s') \in C^r(K, \underline{\pi_r(F)})$ just defined is called the* difference cochain *of the sections $s, s'$.*

**Claim**: $\delta\mathfrak{d}(s, s') = \mathfrak{o}(s') - \mathfrak{o}(s)$.

The proof of this claim is similar to the proof of (2b). Fix an $(r + 1)$-simplex $\Delta_{r+1} =< x_0, \ldots, x_{r+1} >$, restrict the constructions to the sub-complex $\Delta_{r+1}$ of $K$. Under the product structure $E|_{|\Delta_{r+1}} \xrightarrow{\cong} |\Delta_{r+1}| \times E_{x_0}$, we have functions $f, f' : |\Delta^r_{r+1}| \to E_{x+0}$ so that $s(x) = (x, f(x))$ and $s'(x) = (x, f'(x))$. Write simply $\Delta$ for $\Delta_{r+1}$. Form the space $\Delta^r \sqcup_{\Delta^{r-1}} \Delta^r$. We have well defined maps $S^r \to E_{x_0}$

$$S^r = |\Delta^r| \xrightarrow{\iota,\iota'} \Delta^r \bigsqcup_{\Delta^{r-1}} \Delta^r \xrightarrow{d(s,s')} E_{x_0} \tag{20}$$

where $\iota, \iota'$ are the inclusions of the copies of $\Delta^r$ in $\Delta^r \sqcup \Delta^r$ followed by the projection to the identification space. Therefore $f = d(f, f') \circ \iota$ and $f' = d(f, f') \circ \iota'$.

We also have collection of $r + 2$ maps $S^r \to E_{x_0}$

$$S^r = \Delta^r_j \bigsqcup_{\Delta^{r-1}_j} \Delta^r_j \xrightarrow{\iota_j \sqcup \iota_j} \Delta^r \bigsqcup_{\Delta^{r-1}} \Delta^r \xrightarrow{d(f,f')} F \tag{21}$$

where $\iota_j : \Delta^r_j \to \Delta^r$, $j = 0, \ldots, r + 1$ is the inclusion of the face $\Delta^r_j$ opposite the $j^{th}$ vertex of $\Delta$.

Apply now the Hurewicz homomorphism $H$ to both diagrams. In (20) we get the generator of $\pi_r(S^r)$ to go to $H(\mathfrak{o}(s') - \mathfrak{o}(s))$, while the same generator goes to $H(\delta\mathfrak{d}(s, s'))$ in (21). Thus the two obstruction classes are in the same cohomoogy class, proving (2c).

For the proof of (2d) we need to see that given any section $s$ over $K^r$ and $c \in C^r(K, \underline{\pi_r(F)})$ there is a section $s'$ over $K^r$, agreeing with $s$ on $K^{r-1}$, so that $\mathfrak{o}(s') = \mathfrak{o}(s) + \delta c$. in other words, given $s$, let $\mathcal{S}(s) = \{s' : |K^r| \to E|_{|K^r|} : s = s' \text{ on } K^{r-1}\}$. Then

**Claim**: The map $\mathcal{S}(s) \to C^r(K, \underline{\pi_r(F)})$ defined by $s' \to \mathfrak{d}(s, s')$ is surjective.

Recall that for each $r$-simplex $\sigma < K$, $\mathfrak{d}(s, s') \in \pi_r(E_{x_0})$ is defined as follows: let $f, f' = \phi_\sigma \circ s, \phi_\sigma \circ s'$ respectively (in notation of (18)). Identifying $|(\sigma|, |\partial\sigma|) \cong (D^r, S^{r-1})$, define a map $d(f, f') : S^r \to E_{x_0}$ to be $f$ on one hemisphere, $f'$ in the other, well defined on the equator $S^{r-1}$, that is $d(f, f') = f \sqcup f' : S^r = D^r \sqcup_{S^{r-1}} D^r \cong |\sigma| \sqcup_{|\partial\sigma|} |\sigma| \to E_{x_0}$. The surjectivity claim then follows from the following version simplex by simplex:

**Lemma 1.** *Let $S^r = D_1 \cup D_2$ a standard decomposition of the sphere $S^r$ into two disks (hemispheres) intersecting in the equator: $D_1 \cap D_2 = S^{r-1}$. Let $F$ be any space and let $f : D_1 \to F$ and $g : S^r \to F$ be given continuous maps. Then there exists $g' : S^r \to F$ homotopic to $g$ so that $g'|_{D_1} = f|_{D_1}$.*

*Proof.* Since $D_1$ is contractible, we have that $f$ and $g|_{D_1}$ are homotopic, in fact there exists $H_0 : D_1 \times I \cup D_2 \times \{0\}$ so that $H_0(x, 0) = g(x)$ for all $x \in S^r$, and $H_0(x, 1) = f(x)$ for all $x \in D_1$. By the homotopy extension property ($D_1 \times I \cup D_2 \times \{0\}$ is a retract of $S^r \times I$), $H_0$ extends to $H_1 : S^r \times I \to F$. Then $g'(x) = H_1(x, 1)$ is homotopic to $g$ and agrees with $f$ on $D_1$. □

The surjectivity claim follows immediately from this lemma: given any section $s$ and cochain $c \in C^r(K, \underline{\pi_r(F)})$, and take any fixed $r$-simplex $\sigma < K$ and apply the lemma to $f : D_1 \to E_{x_0}$ and $g : S^r \to E_{x_0}$ any representative of $c(\sigma) \in \pi_r(E_{x_0})$. Then $g'|_{D_2}$ represents the $s'_\sigma$.

Finally, the surjectivity claim implies (2d): if $\mathfrak{o}(s) = \delta c$, then there exists $s'$ so that $c = \mathfrak{d}(s, s')$, so $\mathfrak{o}(s) = \delta c = \mathfrak{o}(s') - \mathfrak{o}(s)$ gives $\mathfrak{o}(s') = 0$, in other words, $s'$ extends over $K^{r+1}$.

□

3.1. **The Primary Obstruction.** The class $[\mathfrak{o}(s)]$ of Theorem 5 is called the *primary obstruction* to the existence of a section. We did not state the best theorem concerning $\mathfrak{o}(s)$. In fact, we should have replaced (2c) by the stronger statement *the cohomology class* $[\mathfrak{o}(s)]$ *is independent of* $s$. In view of (1) of Theorem 5, this is equivalent to proving the statement we may call (2c'): if $s$ and $s'$ are sections over $K^r$ that are homotopic over $K^{r-1}$, then $[\mathfrak{o}(s)] = [\mathfrak{o}(s')]$. We will prove this in §3.3.

Assuming this stronger statement, we have the following version of Theorem 5, which is stronger in some ways, less precise in others:

**Theorem 6.** *Let $p : E \to B = |K|$ be a locally trivial fibration with $(r-1)$-connected fiber $F$ as in Theorem 5. Then*

(1) *There are sections $s$ over $K^r$ and any two such sections $s, s'$ are homotopic over $K^{r-1}$*

(2) *Let* $\mathfrak{o}(s)$ *be as in Theorem 5. Then* $\mathfrak{o}(s)$ *is a cocyle and its cohomology class* $[\mathfrak{o}(s)] \in H^{r+1}(K, \underline{\pi_r(F)})$ *is independent of* $s$*. This class is denoted by* $\mathfrak{o}(E)$ *and is called the* primary obstruction *to a section of* $E$*.*
(3) $\mathfrak{o}(E) = 0$ *if and only if* $E$ *has a section* $s$ *defined over* $K^{r+1}$*.*
(4) *If* $L$ *is a complex and* $f : |L| \to |K|$ *is a continuous map, then* $\mathfrak{o}(f^*E) = f^*\mathfrak{o}(E)$*.*

*Proof.* The first three statements are clear from Theorem 5 and the statment (2c') above. For the last statement, may assume, after subdivision, that $f$ is simplicial. Then can find explicit representatives that satisfy the desired identity. □

It is important to understand what this theorem implies and does not imply. First the terminology "primary obstruction" is clear because there are always sections over $K^r$, so it is when we reach $K^{r+1}$ that we first reach an "obstruction". The theorem says that given $s$ over $K^r$ the cohomology class $[\mathfrak{o}(s)] = 0$ if and only if there is a section over $K^{r+1}$, but if $[\mathfrak{o}(s)] = 0$ it does not say that $s$ extends, but only that some $s'$ extends. In this way (2d) of Theorem 5 is much more precise.

3.2. **Back to the Euler class.** The Euler class gives us a good illlustration of Theorems 5 and 6. Going back to the situation of §2, let $\pi : E \to M$ be an oriented vector bundle with fiber $\mathbb{R}^n$ over an $m$-dimensional manifold $M$ triangulated by a complex $K$: $M = |K|$. To have non-trivial situation, assume that $n$ is even and $n \leq m$. For most of the discussion there is no need to assume that the base is a manifold.

We apply the machinery of §3 not to $E$ (which has contractible fibers) but to the bundle $E^* = E \setminus 0 \to M$, where $0$ denotes the zero-section of $E$. Thus the fiber $F = \mathbb{R}^n \setminus 0$ is homotopy equivalent to $S^{n-1}$, it is $(n-2)$-connected and the first non-vanishing homotopy group is $\pi_{n-1}(\mathbb{R}^n \setminus 0) = \pi_{n-1}(S^{n-1}) \cong \mathbb{Z}$. In particular, $r = n - 1$

**Theorem 7.** *Let* $E, E^*, M, F$ *be as just defined. Then*

(1) *The local system* $\underline{\pi_{n-1}(F)}$ *is the constant system* $\mathbb{Z}$*.*
(2) $E^* = E \setminus 0 \to M$ *has a section* $s$ *on the* $(n-1)$*-skeleton* $K^{n-1}$ *of* $K$*, and all sections are homotopic on* $K^{n-2}$*.*
(3) $e(E) = [\mathfrak{o}(s)] \in H^n(M, \mathbb{Z})$*.*
(4) $e(E) = 0$ *if and only if the bundle* $E \to M$ *has a section that does not vanish on the* $n$*-skeleton* $|K^n|$*.*
(5) *In particular, if* $m = n$*,* $E \to M$ *has a nowhere vanishing section if and only if* $e(E)([M]) = 0$*.*

*Proof.* For the first assertion, the local coefficient system $\underline{\mathbb{Z}}$ is determined by the representation $\pi_1(M) \to Aut(\mathbb{Z}) = \{\pm 1\}$ obtained by parallel translation. If $\gamma$ is a loop at the base-point, and the linear transformation $A(\gamma) : \mathbb{R}^n \to \mathbb{R}^n$ is the result of parallel translation in $E$ along $\gamma$, then the automorphism of $\pi_{n-1}(\mathbb{R}^n \setminus 0)$ that determines $\underline{\pi_{n-1}(F)}$ is multiplication by the degree of $A$, which is the same as the determinant of $A$. Since $E$ is oriented, this determinant is always one, hence we get the constant coefficient system $\mathbb{Z}$.

The second assertion follows from (2a) of Theorem 5. For each $n$-simplex $\sigma < K$, $\mathfrak{o}(s)(\sigma) = \deg(f_\sigma)$ where $f_\sigma = h^F_\sigma \circ s$ with $h^F_\sigma$ as in (12). Extending $f_\sigma$ radially, we can regard it as a map $f_\sigma : (|\sigma|, |\partial\sigma|) \to (\mathbb{R}^n, \mathbb{R}^n \setminus 0)$. Looking at the proof of Proposition 5 thus $\deg(f_\sigma : \partial\sigma \to \mathbb{R}^n \setminus 0)$ is the same as the evaluation of $s^*\mu \in H^n(\sigma, \partial\sigma)$, where $\mu$ is the Thom class, on the fundamental homology class of the pair. Thus $\mathfrak{o}(s)(\sigma) = e(E)(\sigma)$ and (3) follows. The fourth and fifth statements are then immediate consequences of Theorem 5.

□

**Remark 2.** We could also consider *non-orientable* vector bundles $E \to M$ with fiber $\mathbb{R}^n$. In this case the first part of Theorem 7 would read: The system $\underline{\pi_{n-1}(F)}$ is the *orientation system*. By definition, this is the system corresponding to the following representation of $\pi_1(M)$: Fix a basepoint. For each loop $\gamma$ at the basepoint, let $A(\gamma) : \mathbb{R}^n \to \mathbb{R}^n$ be "parallel translation" along $\gamma$. The linear transformation $A(\gamma)$ depends on $\gamma$, no just its homopopy class relative to the endpoints. But the sign of the determinant $\det(A(\gamma))$ depends just on the homotopy class of the loop $\gamma$. This is the representation that defines the orientation system, let's denote it $\varepsilon(E)$ It is then reasonable to define the *Euler class* $e(E) \in H^n(M, \varepsilon(E))$ to be the obstruction class.

3.2.1. *Some consequences of Theorem 7.* Applying Theorem 7 to $E = TM$, the tangent bundle of an oriented manifold $M$, we get

**Corollary 1.** *Let $M$ be a closed oriented manifold. Then $M$ has a nowhere vanishing vector field if and only if $\chi(M) = \sum(-1)^i \dim(H^i(M, \mathbb{R})) = 0$.*

*Proof.* Clear. □

What happens when the base is of strictly larger dimension than $n =$ the dimension of the fibre? The fourth statement of Theorem 7 gives us the answer. Perhaps it is more geometric to state this part of the theorem as follows. Assume the base $M$ is a closed *manifold*, and let $K$ be a triangulation: $|K| = M$. Every $\sigma < K$ has a *dual cell* $D(\sigma)$ of complementary dimension and meeting $\sigma$ at exactly one point, their common barycenter. The correspondence $\sigma \to D(\sigma)$ is incidence reversing. The complement of the $\ell$-skeleton $K^\ell$ deformation retracts to the $(m - \ell - 1)$-skeleton of $K^*$. Let us call this the $(m - \ell - 1)$-*co-skeleton* of $M$. Then we can rephrase part (4) of Theorem 7 as follows:

**Corollary 2.** *Suppose $E \to M$ is an oriented vector bundle with fiber $\mathbb{R}^n$ over the closed manifold $M$ of dimension $m > n$. Then $e(E) = 0$ if and only if $E$ has a section $s$ that does not vanish anywhere in the complement of the $(m - n - 1)$-coskeleton of $M$.*

It is difficult, in general, to go beyond this statement in giving conditions for the existence of a nowhere vanishing section of $E$. Suppose, for example, $m = n + 1$ and that $e(E) = 0$. Then we can proceed with the pattern of obstruction theory:

(1) Choose a section $s$ of $E$ that vanishes nowhere on the $n$-skeleton $K^n$.
(2) Form the obstruction cochain $\mathfrak{o}(s) \in C^{n+1}(K, \underline{\pi_n(\mathbb{R}^n \setminus 0)}) = C^{n+1}(K, \mathbb{Z}/2\mathbb{Z})$ if $n \geq 4$..

(3) If $[\mathfrak{o}(s)] = 0 \in H^{n+1}(K, \mathbb{Z}/2\mathbb{Z})$ then $s$ extends to a nowhere vanishing section on $K^{n+1} = M$
(4) If $[\mathfrak{o}(s)] \neq 0$, $s$ cannot be extended to a nowhere vanishing section over $K^{n+1}$. But maybe some other non-vanishing $s'$ on $K^n$ extends over $K^{n+1}$. Go back, make other choices $s'$ over $K^n$ and see what happens.

We can see the difficulties in applying obstruction theory: the fiber can have more non-vanishing homotopy groups, as in (2): here we are using the fact that $\pi_n(S^{n-1}) = \mathbb{Z}/2\mathbb{Z}$ if $n \geq 4$. These non-vanishing homotopy groups bring further obstructions that have to be computed. But another difficulty, perhaps more serious, is seen in (4): the "higher order obstructions" are not indepedents of previous choices, as opposed to the independence of choices of the primary obstruction, namely (2c) of Theorem 5 no longer holds. If we get $[\mathfrak{o}(s)] \neq 0$ all we get is that this particular $s$ cannot extend. But perhaps there is an $s'$, agreeing with $s$ on $K^{n-1}$, which doesn't vanish on $K^n$, and extends over $K^{n+1}$. So we have to check other possibilities.

There is, however, one case where obstruction theory gives a complete answer:

**Proposition 8.** *Suppose $E \to M$ is an oriented vector bundle with fiber $\mathbb{R}^2$. Then $E$ has a nowhere vanishing section if and only if $e(E) = 0 \in H^2(M, \mathbb{Z})$*

*Proof.* By Theorem 7 we see that $E$ has a section $s$ that does not vanish at an point of $K^2$. The obstruction of extending this section to $K^3$ has coefficients in $\pi_2(\mathbb{R}^2 - 0) = \pi_2(S^1) = 0$, so $s$ extends over $K^3$. Since $\pi_i(S^1) = 0$ for all $i > 1$, can continue in this way to find a nowhere vanishing section $s$.

□

The basic principle behind this Proposition is that if the fiber is an *Eilenberg-McLane space*, that is, $\pi_i(F) = 0$ for $i \neq r$, then the primary obstruction in $H^{r+1}(M, \underline{\pi_r(F)})$ determines everything. In fact, we can strengthen Proposition 8 as in the next proposition. Another illustration of this principle is Theorem 8.

**Proposition 9.** *Let $E_1, E_2 \to M$ be oriented rank two vector bundles. Then $E_1$ and $E_2$ are isomorphic if and only if $e(E_1) = e(E_2) \in H^2(M, \mathbb{Z})$.*

*Proof.* We sketch two versions of the proof:

(1) Familiar proof: an oriented rank two vector bundle has transition functions in $GL^+(2, \mathbb{R})$, the group of non-singular 2 by 2 real matrices. Since the inclusion of the rotation group $SO(2) \cong S^1 \to GL(2, \mathbb{R})$ and $S^1 \to \mathbb{R}^2 \backslash 0$ is a homotopy equivalence, the problem is the same as classification of principal $S^1$-bundles over $M$, which is easily obtained by the "exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{C}^\infty(\mathbb{R}) \overset{\exp(2\pi i \cdot)}{\longrightarrow} \mathcal{C}^\infty(S^1) \to 0$$

of sheaves over $M$, where the first is the constant sheaf, the other two are sheaves of germs of the indicated functions. Passing to cohomology we get

$$\cdots \to 0 = H^1(M, \mathcal{C}(\mathbb{R})) \to H^1(M, \mathcal{C}(S^1)) \xrightarrow{\delta_*} H^2(M, \mathbb{Z}) \to 0$$

thus the connecting homomorphism $\delta_*$ is an isomorphism between the group $H^1(M, \mathcal{C}(S^1))$ of isomorphism classes of $S^1$-bundles and the group $H^2(M, \mathbb{Z})$.

(2) A version of the proof using definition and properties of the primary obstruction class in a more direct way would be as follows: Given $E_1, E_2 \to M$, the problem is to find a bundle isomorphism $A : E_1 \to E_2$. This is a section of the bundle $Iso(E_1, E_2) \to M$ with fiber $Iso(\mathbb{R}^2, \mathbb{R}^2)$ of linear isomorphisms $\mathbb{R}^2 \to \mathbb{R}^2$. Since the fibre is isomorphic to $GL^+(2, \mathbb{R})$ and thus homotopy equivalent to the Eilenberg-McLane space $S^1 = K(\mathbb{Z}, 1)$, a section exists if and only if the obstruction class $[\mathfrak{o}(A)] \in H^2(M, \mathbb{Z})$ is zero, where $A$ is a section of $Iso(E_1, E_2)$ over $K^1$.

To compute this class, let $s_1$ be a section of $E_1 \setminus 0$ over $K^1$. Then $s_2 = As_1$ is a section of $E_2$ over $K^1$ and from the definition of the obstruction cochain we see that $\mathfrak{o}(s_2) = \mathfrak{o}(As_1) = \mathfrak{o}(A) + \mathfrak{o}(s_1) \in C^2(K, \mathbb{Z})$: given a 2-simplex $\sigma < K$, $\mathfrak{o}(As_1)(\sigma) = \deg(A \circ s_1)|_{\partial\sigma} = \deg(A|_{\partial\sigma}) + \deg(s_1|_{\partial\sigma})$. Here we use the local trivializtions on $\sigma$ to get maps to the fiber. All fibers are homotopically $S^1$, and we also use that the degree of a pointwise prouct of two maps $S^1 \to S^1$ is the sum of their degrees.

Using the independence of the cohomology class of the primary obstruction from the representative, we get $[\mathfrak{o}(A)] = [\mathfrak{o}(s_2)] - [\mathfrak{o}(s_1)] = e(E_2) - e(E_1) \in H^2(M, \mathbb{Z})$. Thus an isomorphism exists over $K^2$, therefore over $M$, if and only if $e(E_1) = e(E_2)$.

□

3.3. **The difference cochain revisited.** In Definition 4 we gave a definition of the difference cochain $\mathfrak{d}(s, s')$ of two sections $s, s'$ that agree on $K^{r-1}$. This seemed natural in the context we chose for Theorem 5 and the desire of not introducing further machinery into its proof. A more efficient way,, which we pursue now, would have been to use the structure of the product $K \times I$. This also has wider applicability since it applies to sections homotopic over $K^{r-1}$ (but not necessarily equal), thus to the proof of the statment (2c') of §3.1 needed for the proof of Theorem 6.

Going back to the fibration $E \to |K|$ with $(r-1)$-connected fiber $F$, we know we always have sections over $K^r$ and any two are homotopic over $K^{r-1}$. Let $s_0, s_1$ be sections over $K^r$, we can ask when they are homotopic over $K^r$. This can be formulated in terms of a section over $K \times I$ and its natural cell subdivision with cells $\sigma \times 0, \sigma \times 1, \sigma \times I$, for $\sigma < K$, and with $\partial(\sigma \times I) = \partial\sigma \times I + (-1)^{\dim(\sigma)}\sigma \times \partial I$.

Let $s_0, s_1$ be sections of $E$ over $K^r$, let $E' = E \times I \to K \times I$ be the bundle over $K \times I$ induced from $E \to K$ by the projection $K \times I \to K$, let $h : K^{r-1} \times I \to E$ be a homotopy between $s_0$ and $s_1$, meaning $h(x, 0) = s_0(x), h(x, 1) = s_1(x)$ for all $x \in K^{r-1}$. Such a homotopu is equivalent to a section $S = S(s_0, s_1, h)$ of $E'$ over $(K \times I)^r$ defned by

$$S(s_0, s_1, h)(x, t) = \begin{cases} (s_0(x), 0) & \text{for } (x, 0) \in K^r \times 0, \\ (s_1(x), 1) & \text{for } (x, 1) \in K^r \times 1, \\ (h(x, t), t) & \text{for } (x, t) \in K^{r-1} \times I. \end{cases} \tag{22}$$

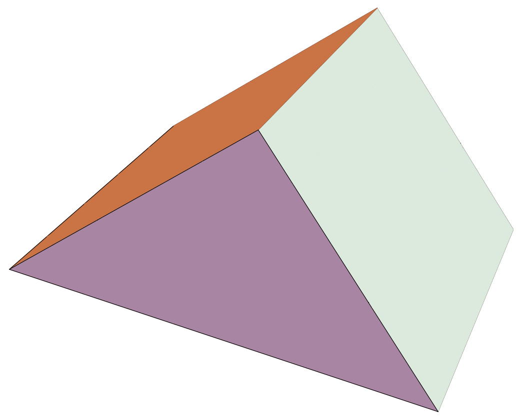

FIGURE 5. $\sigma \times I < K \times I$

Then, arguing with the cell structure of $K \times I$ in the analogous way to our earlier arguments with the cell structure of $K$, and taking care of the local coefficient groups (we will avoid explicit book-keeping), we get an obstruction cochain $\mathfrak{o}(S) \in C^{r+1}(K \times I, \underline{\pi_r(F)})$. Arguing as in the proof of Theorem 5 we get that $\mathfrak{o}(S)$ is a cocycle. The cochains on $K \times I$ split as a direct sum

$$C^k(K \times I) = q^* C^{k-1}(K) \oplus i_0^* C^k(K) \oplus i_1^* C^k(K) \tag{23}$$

where $q : C_k(K \times I) \to C_{k-1}(K)$ is the map on chains defined on basis elements by $q(\sigma \times I) = \sigma$ and $q(\sigma \times 0), q(\sigma \times 1) = 0$. Note that $q$ is *not* a chain map, so the summand $q^* C^{k-1}(K)$ of $C^k(K \times I)$ is not closed under $\delta$ (see Figure 5).

The equation $\delta \mathfrak{o}(S) = 0$ in $C^{r+1}(K \times I, \underline{\pi_r(F)})$ becomes

$$i_1^* S - i_0^* S - \delta q^* d(s_0, s_1, h) = 0 \tag{24}$$

for the cochain $d(s_0, s_1, h) \in C^r(K, \underline{\pi_r(F)})$ with value on the $r$-simplex $\sigma < K$ given (with some abuse of notation) by

$$d(s_0, s_1, h)(\sigma) = [S|_{\partial(\sigma \times I)}] \in \pi_r(F), \tag{25}$$

and $i_0, i_1 : K \to K \times I$ are the two natural embeddings $K \to K \times 0$, $K \to K \times 1$. Thus we see that this obstruction cochain $\mathfrak{o}(S)$ over $K \times I$ contains various cochains we have studied before, and the equation $\delta \mathfrak{o}(S) = 0$ contains several familiar equations. In particular, abusing notation in the same way as in (25), have that

$$\mathfrak{o}(s_0)(\sigma) = [s_0|_{\partial\sigma}] \in \pi_r(F), \quad \mathfrak{o}(s_1)(\sigma) = [s_1|_{\partial\sigma}] \in \pi_r(F), \tag{26}$$

and therefore (24) is equivalent to

$$\mathfrak{o}(s_1) - \mathfrak{o}(s_0) = \delta d(s_0, s_1, h) \tag{27}$$

which is the equation that we need to prove statement (2c') of §3.1, which completes the proof of Theorem 6: If $s, s'$ are sections over $K^r$, they are homotopic over $K^{r-1}$ then $\mathfrak{o}(s)$ and $\mathfrak{o}(s')$ are cohomologous.

Note that (29) gives us that $\mathfrak{o}(s_1) = \mathfrak{o}(s_0)$, then $d(s_0, s_1, h)$ is a *cocycle*. This happens whenever $s_0, s_1$ are sections defined over all of $K$, since, for a section $s$ over $K$, $\mathfrak{o}(s) = 0$. This is needed for the proof of Proposition 10 below.

Moreover, if $s_0 = s_1$ on $K^{r-1}$, and $h = h_0$, the constant homotopy, then, if we let $\mathfrak{d}(s_0, s_1) = d(s_0, s_1, h_0)$, then (24) is the same relation we found in the claim following Definition 4 relating the obstruction cochains and the difference cochain:

$$\delta\mathfrak{d}(s_0, s_1) = \mathfrak{o}(s_1) - \mathfrak{o}(s_0), \tag{28}$$

thus giving us another proof of this equation.

Using these new equations and applying to $S$ and $K \times I$ the same reasoning as in the proof of Theorems 5 and 6 we get:

**Proposition 10.** *Let $s_0, s_1$ be sections of $E$ over $K$ and let $h$ be a homotopy between their restrictions to $K^{r-1}$. Then*

1. *$d(s_0, s_1, h)$ is a cocycle.*
2. *Its cohomolgy class $[d(s_0, s_1, h)] = 0 \in H^r(K, \underline{\pi_r(F)})$ if and only if $s_0$ and $s_1$ are homotopic over $K^r$.*
3. *For all $c \in H^r(X, \overline{\pi_r(F)})$ there exists a section $s$ over $K^r$ such that $s = s_0$ over $K^{r-1}$ and $[\mathfrak{d}(s_0, s)] = c$.*

*Proof.* Apply the proofs of Theorems 5 and 6 to $K \times I$ and the section $S = S(s_0, s_1, h)$. The only point that needs further explanation: in the third statement, previous arguments give $s$ over $K^r$ with $\mathfrak{d}(s_0, s) = c$. Since $c$ is a cocycle, (28) gives $\mathfrak{o}(s) = \mathfrak{o}(s_0) = 0$, thus $s$ extends to $K^{r+1}$

□

3.3.1. *Maps to Eilenberg-McLane spaces.* As an application, recall that, given a finitely generated abelian group $\Pi$ and an integer $n \geq 1$, an *Eilenberg-McLane space* $K(\Pi, n)$ is a space $Z$ with

$$\pi_i(Z) = \begin{cases} 0 & \text{if } i \neq n, \\ \Pi & \text{if } i = n. \end{cases}$$

We have seen the example $S^1 = K(\mathbb{Z}, 1)$, it may be familiar that $\mathbb{C}P^\infty = K(\mathbb{Z}, 2)$, $\mathbb{R}P^\infty = K(\mathbb{Z}/2, 1)$. By general principles it is known that $K(\Pi, n)$ exists and is unique up to homotopy equivalence. But it is rare to find a good model as in the above examples.

The usual proof of existence is: the finitely generated abelian group $\Pi$ is isomorphic to a direct sum $\mathbb{Z}^k \oplus C_1 \oplus \cdots \oplus C_l$ where, for $i = 1, \ldots, l$, $C_i$ is a finite cyclic group of order $d_i > 1$. Forma a CW complex as follows:

1. Take a single 0-cell $e_0$.
2. Take $k + l$ cells of dimension $n$, $e_n^1, \ldots e_n^{k+l}$ and attach them to $e_0$ by the only possible map $\partial e_n^i \to e_0$, $i = 1, \ldots, k + l$.

(3) Take $l$ cells of dimension $n+1$, $e^1_{n+1}, \dots e^l_{n+1}$ and attach $e^i_{n+1}$ to the image of $e^{k+i}_n$ by a map $\partial e^i_{n+1} \to \overline{e^{k+i}_n}$ of degree $d_i$. The result is a space $Z_0$ with $\pi_i(Z_0) = 0$ for $0 \le i < n$ and $\pi_n(Z_0) = \Pi$, but have no knowledge of $\pi_i(Z_0)$ for $i > n$.
(4) Now attach cells of dimension $n+2$ to get $Z_1$ with $\pi_i(Z_1) = 0$ for $0 \le i < n$ and $i = n+1$ and $\pi_n(Z_1) = \Pi$. Then attach $n+3$ cells to get $Z_2$ with $\pi_i(Z_2) = 0$ for $0 \le i < n$ and $n < i \le n+2$ keeping $\pi_n(Z_2) = \Pi$, and so on. Then $\cup_k Z_k$ is a $K(\Pi, n)$.

In particular, we see that $K(\Pi, n)$ can be realized as a $CW$-complex $Z$ with $n-1$-skeleton $Z^{n-1}$ consisting of a single point $p_0$. We will make this assumption from now on.

The following theorem says that, given any finite complex $X$, there is a one to one correspondence between the set $[X, K(\Pi, n)]$ of homotopy classes of maps $f : X \to K(\Pi, n)$ and the cohomology group $H^n(X, \Pi)$. In other words, the space $K(\Pi, n)$ represents the functor $H^n(\,\cdot\,, \Pi)$. More precisely:

**Theorem 8.** *Let $\Pi$ be a finitely generated abelian group and let $n \ge 1$.*

(1) *For any abelian group $A$ there is a natural isomorphism $H^n(K(\Pi, n), A) \cong Hom(\Pi, A)$.*
(2) *Let $\eta \in H^n(K(\Pi, n), \Pi)$ correspond to the identity in $Hom(\Pi, \Pi)$, and let $X$ and $[X, K(\Pi, n)]$ be as above. Then the map $[X, K(\Pi, n)] \to H^n(X, \Pi)$ defined by $f \to f^*\eta$ is a bijection.*

*Proof.* For the first statement, by Hurewicz we have $H_i(K(G, n), \mathbb{Z}) = 0$ for $i < n$ and the natural map $\pi_n(K(\Pi, n) \to H_n(K(\Pi, n), \mathbb{Z})$ is an isomorphism. By the universal coefficient theorem, for any abelian group $A$, we have $H^n(K(\Pi, n), A) \cong Hom(\pi_n(K(\Pi, n)), A) \cong Hom(\Pi, A)$.

For the second statement, note that maps $f : X \to K(\Pi, n)$ correspond to sections $s(x) = (x, f(x))$ of the trivial bundle $X \times K(\Pi, n)$. We can use the obstruction theory for homotopies of sections of Proposition 10: making an obvious change of notation writing $f$ intsead of the section $s$, we see first that, up to homotopy, we may assume that any map $f : X \to K(\Pi, n)$ maps the $(n-1)$-skeleton $X^{n-1}$ to the basepoint $p_0$ (the $(n-1)$-skeleton of the above construciton of $K(\Pi, n)$), so all maps agree on $X^{n-1}$. Proposition 10 imples that $f_0$ is homotopic to $f_1$ over $X^n$ if and only if $[\mathfrak{d}(f_0, f_1)] = 0 \in H^n(X, \underline{\pi_n(K(\Pi, n))}) = H^n(X, \Pi)$.

Now, for each $n$-simplex $\sigma$ of $X$, $\mathfrak{d}(f_0, f_1)(\sigma)$ is represented by the map of the $n$-sphere $\partial(\sigma \times I)$ that agrees with $f_0$ on $\sigma \times 0$, with $f_1$ on $\sigma \times 1$ and maps $\partial\sigma \times I$ to $p_0$ (see Figure 5). This element of $\pi_n(K(\Pi, n))$ is obtained from two maps of the $n$-sphere $\sigma/\partial\sigma$, namely $f_0|_\sigma$ and $f_1|_\sigma$ as their difference in the group $\pi_n(K(\Pi, n)) = \Pi$, in other words,

$$[\mathfrak{o}(f_0, f_1)(\sigma)] = [f_1|_\sigma] - [f_0|_\sigma] \in \Pi. \tag{29}$$

Now, our maps $f : X \to K(\Pi, n)$ factor $X \to X/X^{n-1} \to K(\Pi, n)$, for any $n$-simplex $\sigma$, $H([f|\sigma]) = f_*(\sigma) \in H_n(K(\Pi, n), \mathbb{Z})$, where $H$ is the Hurewicz homomorphism. Then, by definition, $f^*\eta(\sigma) = \eta(f_*\sigma) = [f|_\sigma]$. Thus (29) is

equivalent to
$$\mathfrak{d}(f_0, f_1) = f_1^*\eta - f_0^*\eta,$$
and Proposition 10 gives us that $f_0$ is homotopic to $f_1$ over $X^n$ if and only if $f_0^*\eta = f_1^*\eta$, and every $\alpha \in H^n(X, \Pi)$ can be obtained as $f^*\eta - f_0^*\eta$ for some $f : X^{n+1} \to K(\Pi, n)$. Since $\pi_i(K(\Pi, n)) = 0$ for $i > n$, there are not further obstructions to homotopies between $f_0$ and $f_1$ or to extension of $f$. Thus we can conclude that the map $f \to f^*\eta$ is both injective and surjective as a map $[X, K(\Pi, n)] \to H^n(X, \Pi)$. □

This theorem says that $\eta$ is the "universal" $n$-dimensional cohomology class with coeffcients in $\Pi$.

3.3.2. *Maps to Spheres.* The proof of the preceding theorem can be easily modified to give the following theorem of Hopf:

**Theorem 9.** *Let $M$ be a closed, oriented $n$-dimensional manifold, and let $[M, S^n]$ denote the set of homotopy classes of maps $f : M \to S^n$. Then the map* $\deg : [M, S^n] \to \mathbb{Z}$ *is a bijection.*

*Proof.* In dimensions $\leq n$ it is the same obstruction proof as in Theorem 8, with $\Pi = \pi_n(S^n) = H_n(S^n) = H^n(S^n) = \mathbb{Z}$ and $\eta \in H^n(S^n, \mathbb{Z})$ a generator. Higher obstructions now vanish because $\dim(M) = n$. □

## 4. Stiefel-Whitney Classes

We started our discussion of characteristic classes with the Euler class $e(E)$ of an oriented vector bundle $E \to X$ with fiber $\mathbb{R}^n$. We saw that one possible definition of $e(E)$ :

$e(E) \in H^n(X, \mathbb{Z})$ *is the primary obstruction to finding a nowhere vanishing section of $E$*

This definition makes sense because it asks for a section of the associated bundle $E \setminus 0$ with $(n-2)$-connected fibre $F = \mathbb{R}^n \setminus 0 \sim S^{n-1}$ and $\pi_{n-1}(F) = \mathbb{Z}$, so the primary obstruction is indeed an element of $H^n(X, \mathbb{Z})$.

In the same spirit the *Stiefel-Whitney classes* of $E$ were originally defined as follows:

*For $k = 1, \ldots n$, the $k^{th}$ Stiefel-Whitney class of $E$, $w_k(E)$, is the primary obstruction to finding $n - k + 1$ sections of $E$ that are linearly independent at each point.*

This is a temporary definition to motivate Defiinition 6 below. As it stands, $w_n(E)$ is the primary obstruction to finding $n - n + 1 = 1$ sections of $E$ independent at each point, in other words, a nowhere vanishing section, while $w_1(E)$ is the primary obstruction to finding $n - 1 + 1 = n$ sections linealry indpendent at each point, in other words, a trivialization. Thus, for now, $w_n(E) = e(E)$.

Let's look at $w_1(E)$, the primary obstruction to finding $n$ sections that form a basis at each point. Equivalently, we want to find a section of the bundle of bases of the fibers of $E$. The space of basis of $\mathbb{R}^n$ is homoemorphic to the $GL(n, \mathbb{R})$, the space of iinvertible $n$ by $n$ matrices., which has two connected components given by the sign of the determinant. Thus $w_1(E)$ is the primary obstruction to a section of $GL(E)$, the bundle associated to $E$ with fiber $GL(n, \mathbb{R})$.

We are in the situation of the first step of the procedure of §3: $GL(E)$ has a section $s$ over the 0-skeleton, it can be be extended to a section over the one-skeleton if and only if, for every one-simplex $< x_0, x_1 >$, $\det(s(x_0))$and $\det(s(x_1))$ have the same sign. We are in the situation we previously avoided of considering $\pi_0$, which is, in general, not a group. but, in this case it is the multiplicative group $\{\pm 1\}$, or, in additive notation, $\mathbb{Z}/2$ that is, if and only if the cochain $\mathfrak{o}(s) \in C^1(K, \mathbb{Z}/2)$ defined by $\mathfrak{o}(s) < x_0, x_1 >= 0$ respectively 1 if $\det(s(x_0), \det(s(x_1))$ have the same, respectively opposite signs, is the zero cochain. Then the general reasoning of Theorem 5 gives us that $GL(E)$ has a section over the one-skeleton if and only if $[\mathfrak{o}(s)] = 0 \in H^1(K, \mathbb{Z}/2)$.

Over the one-skeleton having a seciton of $GL(E)$ is the same as having a section of SignDet($E$), the associated bundle with fiber $\{\pm 1\}$ given by the sign of the determiant. While the section $s$ over the one-skeleton may not extend any further, there are no further obstructions to extending SignDet($s$). Thus we see that the interpretaion of $w_1(E) = [\mathfrak{o}(s0]$ is:

(30) $\quad w_1(E) \in H^1(X, \mathbb{Z}/2)$ and $w_1(E) = 0$ if and only if $E$ is orientable.

Now let's see the interpretation of the classes $w_k(E)$ for $0 < k < n$. They are primary obstructions to sections of associatied *Stiefel bundles* $St_l(E)$ with fiber the *Stiefel manifold* $St_l(\mathbb{R}^n)$ *of l-frames in* $\mathbb{R}^n$, namely the manifold of collections of $l$ linearly independent vectors $\{v_1, \dots, v_k l\}$ in $\mathbb{R}^n$. Since we are only interested in the homotopy type of the fiber we may assume that $\{v_1, \dots, v_l\}$ form an orthonormal set, since the Gram-Schmidt process deformation retracts the space $St_l(\mathbb{R}^n)$ of independent $l$-tuples to the space of orthonormal ones, which we will denote $V_{n,l}$:

**Definition 5.** *If* $1 \leq l \leq n$, *the* Stiefel manifold of orthonormal $l$ frames in $\mathbb{R}^n$ *is the submanifold* $V_{n,l}$ *of* $\mathbb{R}^{ln}$ *defined by*

$$V_{n,l} = \{v_1, \dots, v_l : v_1, \dots v_l \in \mathbb{R}^n \text{ and } v_i \cdot v_j = \delta_{i,j} \text{ for all } i, j\}$$

Here $l$ is a function of $k$ (and $n$), to be determined by the requirement that $V_{n,l}$ be $(k-1)$-connected. Thus we need to compute the first non-vanishing homotopy group of $V_{n,l}$. Leaving the case of $V_{1,1} = \{\pm 1\}$ aside, the result is:

**Theorem 10.** *For* $n \geq 2$ *we have* $\pi_i(V_{n,l}) = 0$ *if* $i \leq n-l-1$ *and*

$$\pi_{n-l}(V_{n,l}) = \begin{cases} \mathbb{Z} & \text{if } l = 1 \text{ or } n-l \text{ is even }, \\ \mathbb{Z}/2 & \text{otherwise.} \end{cases}$$

*Proof.* First note that we just treated, in the discussion of $w_1$, the space $V_{n,n} = O(n)$, the orthogonal group, which has two connected components, thus $\pi_0(V_{n,n}) = \mathbb{Z}/2$ is the first non-vanishing homotopy group. So assume $l < n$ and first note that if $v_1, \dots, v_l$ is orthonormal, there is a rotation matrix $R \in SO(n)$ such that $v_i = Re_i$ for $i = 1, \dots, l$, where $\{e_i : i = 1, \dots n\}$ is the standard basis of $\mathbb{R}^n$. Thus, for $l < n$, $V_{n,l}$ is a homogeneous space for the connected group $SO(n)$, hence it is connected. The subgroup that leaves $e_1, \ldots, e_l$ fixed is the subgroup, isomorphic to $SO(n-l)$ of rotations that are the identity on the first summand of $\mathbb{R}^n = \mathbb{R}^l \oplus \mathbb{R}^{n--l}$, and arbitrary on the second summand. Thus we have

$$V_{n,l} = SO(n)/SO(n-l) \text{ for } 1 \leq l < n. \tag{31}$$

If $l = 1$ then $V_{n,1} = S^{n-1}$, thus the first non-vanishing homotopy group is as asserted in the theorem. So let's assume $1 < l < n$. Then we have two natural maps;

(1) A fibration $p_l : V_{n,l} \to V_{n,l-1}$ taking $v_1, \ldots, v_l$ to $v_1, \ldots, v_{l-1}$
(2) An injective map $\iota_l : S^{n-l} \to V_{n,l}$ taking $v$ in the unit sphere of the second summand of $\mathbb{R}^n = \mathbb{R}^{l-1} \oplus \mathbb{R}^{n-l+1}$ to the frame $e_1, \ldots, e_{l-1}, v$ in $V_{n,l}$.

Observe that the image of $\iota_l$ is the fiber of $p$ over the point $\{e_1, \ldots, e_{l-1}\} \in V_{n,l-1}$.

In other words, we have a fibration

$$\begin{tikzcd} S^{n-l} \arrow[r, "\iota_l"] & V_{n,l} \arrow[d, "p_l"'] \\ & V_{n,l-1}. \end{tikzcd} \tag{32}$$

The exact homotopy sequence of this fibration

$$\ldots \longrightarrow \pi_{i+1}(V_{n,l-1}) \xrightarrow{\partial_*} \pi_i(S^{n-l}) \xrightarrow{(\iota_l)_*} \pi_i(V_{n,l}) \xrightarrow{(p_l)_*} \pi_i(V_{n,l-1}) \longrightarrow \ldots \tag{33}$$

easily gives, by induction on $l$,

**Lemma 2.** (1) *$\pi_i(V_{n,l}) = 0$ for $i < n-l$.*
(2) *$\pi_{n-l}(V_{n,l})$ is generated by the homotopy class of the map $\iota_l : S^{n-l} \to V_{n,l}$*

In particular, we get that $\pi_{n-l}(V_{n,l})$ is a cyclic group and we get a very concrete geometric description of a generator, namely the most obvious way of getting a sphere of $l$-frames from a sphere of unit vectors..

It remains to determine the order of this cyclic group. To this end, look again at the homotopy sequence (33) for $i = n-l$. We get

$$\pi_{n-l+1}(V_{n,l-1}) \xrightarrow{\partial_*} \pi_{n-l}(S^{n-l}) \xrightarrow{(\iota_l)_*} \pi_{n-l}(V_{n,l}) \longrightarrow 0 \tag{34}$$

The middle group is isomorphic to $\mathbb{Z}$ and the group we want is $\mathbb{Z}$ modulo the image of the connecting homomorphism $\partial_*$

To do this, we have to compute the connecting homorphism. Recall how $\partial_*([f])$ is defined on the homotopy class of a map $f : S^m \to B$ of the base space of a fibration $F \to E \to B$: represent $[f]$ by a map (still called) $f$, of the disk, $f : D^m \to B$ taking $\partial D^m$ to the basepoint in $B$. Lift this to $\tilde{f} : D^m \to E$, then $\tilde{f}|_{\partial D^m}$ maps $\partial D^m$ to the fiber $F$ over the basepoint. This map $\tilde{f}|_{\partial D^m}$ is a map $S^{m-1} \to B$ representing $\partial_*([f])$.

Let us apply this to the fibration (32). Since we know, from Lemma 2, that $\pi_{n-l+1}(V_{n,l-1})$ is generated by the homotopy class of the map $\iota_{l-1} : S^{n-l+1} \to V_{n,l-1}$, we have to find $\partial_*([\iota_{\tilde{l-1}}])$. Following the above procedure, the map $S^{n-l} \to S^{n-l}$ representing $\partial_*([\iota_{\tilde{l-1}}])$ that we get from the fibration (32) is the same as the map that we obtain by resticiting this fibration to the image of $\iota_{l-1}$.

The map $S^{n-l} \to S^{n-l}$ representing $\partial_*([\widetilde{\iota_{l-1}}])$ that we get from the above procedure involves only the part of the ibration (32) that lies over the image of $\iota_{l-1}$, in other words, we can replace (32) by the fibration $\iota_{l-1}^* V_{n,l}$ over $S^{n-l+1}$ induced from it by $\iota_{l-1}$.

Recalling that $\iota_{l-1}$ sends the unit vector $v$ in the second summand of $\mathbb{R}^n = \mathbb{R}^{l-2} \oplus \mathbb{R}^{n-l+2}$ to the $(l-1)$-frame frame $e_1, \dots, e_{l-2}, v$, we see that all frames involved in the fibration $\iota_{l-1}^* V_{n,l}$ contain the fixed $(l-2)$-frame $e_1, \dots e_{l-2}$. Thus this induced fibration is isomorphic to the fibration

(35)
$$\begin{tikzcd} S^{n-l} \arrow[r] & V_{n-l+2,2} \arrow[d] \\ & S^{n-l+1} \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} S^{m-2} \arrow[r] & V_{m,2} \arrow[d] \\ & S^{m-1}. \end{tikzcd}$$

by the map that assigns to each frame $f$ the frame $e_1, \dots, e_{l-2}, f$. In particular, the image of $\partial_*$ in (34) is the same subgroup of $\pi_{n-l}(S^{n-l}) \cong \mathbb{Z}$ as the image of the connecting homomorphism for (35).

We rewrote the fibration using $m = n - l + 2$ in order to simplify notation in the next lemma:

**Lemma 3.** (1) *The fibration (35) is the same as the fibration of the unit tangent bundle $T^1 S^{m-1}$ of $S^{m-1}$.*
(2) *For the fibration $S^{m-2} \to T^1 S^{m-1} \to S^{m-1}$, the image of the connecting homomorphism is $(1 + (-1)^{m-1})\mathbb{Z} \subset \mathbb{Z}$.*

*Proof.* For the first statement, a tangent vector at $x \in S^{m-1} \subset \mathbb{R}^m$ is a vector $y \in \mathbb{R}^m$ perpendicular to $x$, thus $V_{m,2} = T^1 S^{m-1}$.

For the second assertion, use the procedure of computing the connecting homorphism $\pi_{m-1}(S^{m-1}) \to \pi_{m-2}(S^{m-2})$ in the unit tangent bundle fibration: take a map $f : D^{m-1} \to S^{m-1}$ that is diffeomorphic onto its image in the interior of $D^{m-1}$ and sends $\partial D^{m-1}$ to a point. A lift $\tilde{f} : D^m \to T^1 S^{m-1}$ is a unit tangent vector field to $S^{m-1}$ over the complement of the basepoint, converging to a mapping of $\partial D^m$ to the fiber over the basepoint. By suitable scaling in the tangent bundle this is equivalent to a vector field on $S^{m-1}$ with a single zero at the basepoint, and with the local index at the basepoint generating the image of the connecting homomorphism in $\pi_{m-2}(S^{m-2}) \cong \mathbb{Z}$. By the Poincaré - Hopf theorem this index is the same as $\chi(S^{m-1}) = 1 + (-1)^{m-1}$.

□

Putting all this together, we see that for $1 < l \leq n$, $\pi_{n-l}(V_{n,l}) = \mathbb{Z}/(1 + (-1)^{n-l-1})\mathbb{Z}$, finishing the proof of the theorem.

□

With this information, we can finish the definition of the Stiefel-Whitney classes $w_k(E)$. If it iw to be a primary obstruction to findinf $l$ sections linealy independent at each point, then Theorem 10 tells us, first of all, that $k = n - l + 1$, that is, $l = n - k + 1$ as we had asserted earlier. Moreover, the local coefficient system $\underline{\pi_{n-k+1}(St_{n-k+1}(\mathbb{R}^n))}$ has fiber either $\mathbb{Z}$ or $\mathbb{Z}/2$. In either case there is a *unique* non-zero homomorphism

$$r : \underline{\pi_{n-k+1}(St_{n-k+1}(\mathbb{R}^n))} \to \mathbb{Z}/2 \tag{36}$$

(where $\mathbb{Z}/2$ is the constant coefficient system) , namely reduction modulo 2.

Finally, here's the definition:

**Definition 6.** *Let $E \to X$ be a vector bundle with fiber $\mathbb{R}^n$, let $1 \leq k \leq n$, and let $\mathfrak{o}_k(E)$ be the primary obstruction to finding $n-k+1$ sections of $E$ that are linearly independent at each point $x \in X$. Then the $k$-th* Stiefel - Whitney class of *$E$, $w_k(E) \in H^k(X, \mathbb{Z}/2)$ is defined by $w_k(E) = r(\mathfrak{o}_k(E))$, where $r$ is reduction modulo 2 as in (36).*

## References

[1] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*,
[2] J. W. Milnor and J. D. Stasheff, *Characteristic Classes*,
[3] N. Steenrod, *The Topology of Fiber Bundles*,

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