Math 5210, Review of Definitions and Theorems

Should be familiar with Chapters 1 to 4 and parts of Chapter 7 of Rudin, and the notes. Here is a list of some of the more important topics. (References: R = Rudin, N = Notes)

Definitions

1. Definition of $\mathbb{R}$ as equivalence classes of Cauchy sequences of rational numbers (N §2). Know the equivalence relation, how to define the field operations and the order.
2. Completeness of the real numbers, infimum, supremum (R Chap 1).
3. Metric Spaces (R Chap 2). Examples of metric spaces: $\mathbb{R}$, $\mathbb{R}^k$ (several disstance functions d_1, d_2, d_∞ (N 3.1.2)), the space $(C(X), d_\infty)$ of continuous functions (R 7.14, N 3.1.3)
4. Basic terminology: neighborhood, open sets, closed sets, etc (all items of R 2.18), also closure and interior. Compact sets, connected sets.
5. Convergent sequences, Cauchy sequences in a metric space (R Chap 3). $\limsup$ and $\liminf$ for real sequences. Complete metric space (N 3.1, R 3.12).
6. Continuous map $f : X \to Y$ (R Chap 4).
7. Lipschitz map (N 3.4), contraction (R 9.22).
8. Uniform convergence of sequences of functions, uniformly Cauchy sequences (R 7.7 to 7.9)

Theorems

1. d_1, d_2, d_∞ are metrics on $\mathbb{R}^k$ (know how to check this, for example triangle inequality). Know how to prove that these metrics are complete.
2. If X is a compact metric space, d_∞ on $C(X)$ is a metric. This metric is complete (N 3.1.2, R 7.15) .
3. $f : X \to Y$ continuous. Then X compact $\Rightarrow$ $f(X)$ is compact; X connected $\Rightarrow$ $f(X)$ is connected
4. Compact sets are closed. Compact sets are bounded. Converse need not be true.
5. A closed subset of a compact set is compact. An infinite subset of a compact set has limit points. Finite intersection property of compact sets (R 2.36).
6. Intervals in $\mathbb{R}$ are connected (R 2.47) Convex sets in $\mathbb{R}^k$ are connected (R chp 2, ex 21).
7. f differentiable and $[f'(x)| \leq C$ (or $||df|| \leq C$) $\Rightarrow$ f is Lipschitz. (N 3.4)
8. X compact, $f : X \to \mathbb{R}$ continuous $\Rightarrow$ f attains its maximum and minimum.
9. Contraction mapping theorem (R 9.23). Application to Newton's method and solutions of ODE's. (N §4).