1. Let $1 \leq p < \infty$ and recall that we have defined the p-norm on the space $C[0,1]$ of continuous functions on $[0,1]$ by

$$||f||_p = (\int_0^1 |f(x)|^p dx)^{\frac{1}{p}}$$

Prove that $C[0,1]$ is *not* complete in this norm. *Suggestion*: You may want to look at the proofs in §3.2 of the notes that $C[0,1]$ is not complete in $||f||_1$.

2. Let A be the set of rational numbers in $[0,1]$ and let $\{I_i\}_{i=1}^n$ be a *finite* collection of open intervals covering A. Prove that

$$\sum_{i=1}^{n} |I_i| \geq 1.$$

3. Let X be any set. Prove that the following two $[0,\infty]$-valued functions on 2^X give measures defined on the σ-algebra 2^X:

   (a) The *counting measure* $\mu_c : 2^X \to [0,\infty]$ defined by $\mu_c(A) = n$ if A is finite of cardinality n, and $\mu_c(A) = \infty$ otherwise.

   (b) Fix $x_0 \in X$, define the *point mass* measure concentrated at x_0, $\mu_{x_0} : 2^X \to \mathbb{R}$ denoted by $\mu_{x_0}(A) = 1$ if $x_0 \in A$ and $\mu_{x_0}(A) = 0$ otherwise.

4. Pugh, Appendix A, gives an example of a non-measurable set in $[0,1) \subset \mathbb{R}$ based on the irrational rotations of the circle. As far as measure theory is concerned, the interval $[0,1)$ is the same as the circle $C = \{z \in \mathbb{C} : |z| = 1\}$ by the bijection $[0,1) \to C$ that takes x to $e^{2\pi i x}$. This is a continuous bijection that is not a homeomorphism, but but measure theory does not see the discontinuity in the inverse. It is more natural to construct the non-measurable set in C, and then transfer it to $[0,1)$.

   Read Appendix A through the end of the proof of theorem 45, then write down in detail why the following statements in the beginning of the proof are true:

   (a) The orbits of R are disjoint sets,

   (b) there are uncountably many of them,

   (c) and they divide the circle as $C = \coprod_{n\in\mathbb{Z}} R^n(P)$.