1. *Completion of a metric space* We have seen how useful it is to have a *complete* metric space in order to have solutions to analytical problems, for example, fixed points of contractions. One fact to keep in mind is that any metric space can be enlarged to a complete metric space, call its *completion.* A familiar example is $\mathbb{Q}$ can be completed to $\mathbb{R}$, in other words, $\mathbb{R}$ is the completion of $\mathbb{Q}$

   One way to complete a metric space is to use the same process as the construction of $\mathbb{R}$ from $\mathbb{Q}$ as the collection of Cauchy sequences in $\mathbb{Q}$ modulo an equivalence relation. This takes a certain amount of verification, see, for example, the notes for this course, or problem 24 of chapter 3 of Rudin.

   A quicker way is to use the method of problem 24, chapter 7 of Rudin. Since we have been using the symbol $\mathcal{C}(X)$ with a different meaning, some explanation is in order.

   If X is *any* metric space, not necessarily compact, we can define a space $BC(X)$ of *bounded continuous functions* on X:

   $$BC(X) = \{f : X \to \mathbb{R} : f \text{ is bounded } \}$$

   with distance function

   $$d_\infty(f, g) = \sup\{|f(x) - g(x)| : x \in X\}..$$

   If X is compact this is the same as the space we called $C(X)$ before. For any X, this is the space Rudin calls $\mathcal{C}(X)$. The same proof that we used for completeness of $C(X)$ works to show that for *any* metric space, $BC(X)$ is *complete*, see theorem 7.15 of Rudin.

   With this understood, do exercise 24 of chapter 7 of Rudin: Given any metric space X, fix a point $a \in X$ and assign to each $p \in X$ the function f_p defined by

   $$f_p(x) = d(x, p) - d(x, a).$$

   Prove that $f_p \in BC(X)$, $d_\infty(f_p, f_q) = d_X(p, q)$. Then the map $\Phi : X \to BC(X)$ is an *isometry* and the closure of $\Phi(X)$ is a completion of X.

2. Rudin, Chapter 7, 15, 18, 20, 23.