1. Rudin, Chapter 9, Problem 5.

2. *Tangent Planes* The implicit function theorem tells us: Suppose $U \subset \mathbb{R}^{n+1}$ is an open set, $g : U \to \mathbb{R}$ is of class C^1 and $p = (p_0, r) \in U$ ($p_0 \in \mathbb{R}^n, r \in \mathbb{R}$) is such that $g(p) = 0$ and $\frac{\partial g}{\partial x_{n+1}}(p) \neq 0$. Then there are neighborhoods N of p and N_0 of p_0 and a C^1 function $\phi : \mathbb{N}_0 \to \mathbb{R}$ so that

$$Z = \{q \in N : g(q) = 0\} = \{(x_1, \dots, x_n, \phi(x_1, \dots, x_n)) : (x_1, \dots, x_n) \in N_0\},$$

in other words, *locally* Z can be represented in two equivalent ways: as the zero set of g or as the graph of ϕ. Let $\Phi : N_0 \to \mathbb{R}^{n+1}$ (a parametric representation of the graph of ϕ) be defined by

$$\Phi(x_1, \dots, x_n) = (x_1, \dots, x_n, \phi(x_1, \dots, x_n)).$$

In advanced calculus (for $n = 2$, but the same works for any n) the tangent plane to Z at p, which we denote by T_pZ, is defined in two different ways:

- As the image of $d_{p_0}\Phi : \mathbb{R}^n \to \mathbb{R}^{n+1}$, that is, $T_pZ = \{d_{p_0}\Phi(v) : v \in \mathbb{R}^n\}$.
- As the kernel of $d_pg : \mathbb{R}^{n+1} \to \mathbb{R}$, that is, $T_pZ = \{w \in \mathbb{R}^{n+1} : d_pg(w) = 0\}$.

Show that these two spaces are the same, therefore the two definitions are equivalent.

3. *Lagrange Multipliers* With the same notation as in the last problem, suppose $f : U \to \mathbb{R}$ is of class C^1. We say that $p \in Z$ is a *critical point of* $f|_Z$ if and only if p_0 is a critical point of $f \circ \Phi$. Prove

(a) p is a critical point of $f|_Z$ if and only if $d_pf(v) = 0$ for all $v \in T_pZ$.

(b) p is a critical point of $f|_Z$ if and only if there exists $\lambda \in \mathbb{R}$ such that

$$d_pf = \lambda d_pg, \quad g(p) = 0.$$

4. Apply the method of Lagrange multipliers to prove the following: suppose $A = \{a_{i,j}\}$ is a *symmetric* n by n matrix, let $f(x)$ be the associated quadratic form:

$$f(x_1, \dots, x_n) = \sum_{i,j=1}^{n} a_{i,j}x_ix_j = Ax \cdot x = x^tAx$$

Let S be the unit sphere in $\mathbb{R}^n$. Prove that $x \in S$ is critical for $f|_S$ if and only if x is an eigenvector of A. (This means, there exists $\lambda \in \mathbb{R}$ so that $Ax = \lambda x$), and that the value of f at a critical point is the corresponding eigenvalue λ.

Suggestion: Prove that $d_xf = 2Ax$ by showing that for $x, v \in \mathbb{R}^n$ and $h \in \mathbb{R}$, $f(x + hv) = f(x) + 2hv^tAx + O(h^2)$. You may need to use $v^tAx = (v^tAx)^t = x^tA^tv$ and $A = A^t$.

5. Rudin Chapter 9, Problems 9, 10.