1. Let V and W be finite dimensional normed vector spaces. Let $L(V,W)$ denote the vector space of all linear transformations from V to W. If $A \in L(V,W)$, define the *operator norm* of A to be

$$||A|| = \sup\{||Av||_W : v \in V, ||v||_V = 1\} \ (= \sup\{||Av||_W/||v||_V : v \in V, v \neq 0\})$$

in other words, $||A||$ is the maximum value of $||Av||_W$ over the unit sphere in V. Theorem 9.7 in Rudin shows that this is indeed a norm on $L(V,W)$. Moreover, if U, V, W are normed vector spaces, $A \in L(U,V)$, $B \in L(V,W)$, then $||BA|| \leq ||B||||A||$.

Suppose $V = \mathbb{R}^m$, $W = \mathbb{R}^n$ with the usual norms $||v|| = \sqrt{v \cdot v}$ were $u \cdot v$ is the usual dot product. Elements of $L(V,W)$ are in one-to-one correspondence with m by n matrices, we will identify transformations with matrices via the standard basis or $\mathbb{R}^m, \mathbb{R}^n$. See equation (6) of Chapter 9 of Rudin for the useful estimate

$$||A|| \leq (\sum_{i,j} a_{ij}^2)^{\frac{1}{2}}.$$

Exercise: Prove that for $A \in L(\mathbb{R}^m, \mathbb{R}^n)$, $||A|| = \sqrt{\lambda_{\max}}$ where $\lambda_{\max}$ is the largest eigenvalue of the symmetric positive semi-definite matrix $A^t A$.

2. Rudin Chapter 9, Problems 16, 17, 18, 19.