1. Let $\phi : [a,b] \to \mathbb{R}$ be a continuously differentiable function. Suppose there is an interval $[c,d] \subset (a,b)$ so that $\phi([c,d]) \subset [c,d]$ and suppose there is a constant $C < 1$ so that $|\phi'(x)| < C$ on $[c,d]$. Then $\phi$ is a contraction on the complete metric space $[c,d]$, so it has a unique fixed point on $[c,d]$ that is an attractive fixed point and can be found by iteration.

   The purpose of this exercise is to apply this reasoning to the Newton map associated to a twice continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$, to see that these works on such intervals, and also to see what can happen outside such intervals.

   Recall that the Newton map $N$ is the map defined for $f'(x) \neq 0$ and given by

$$N(x) = x - \frac{f(x)}{f'(x)}.$$

   The fixed points of $N$ are the zeros of $f$.

   (a) Let $f(x) = x^2 - 2$. Find an explicit interval containing each of the two fixed points $\pm\sqrt{2}$ of $N$ where there is a constant $C < 1$ so that $|N'(x)| < C$ and the reasoning of Problem (1) applies. Pick a point in this interval, iterate $N$, and see what happens. (See Figure 5.1 of the notes, reproduced as Figure 1 on the next page).

   (b) Now iterate $N$ beginning at $x = 0.1$ and $x = 0.001$ where clearly $|N'(x)| > 1$, and see what happens.

   (c) Do the same for the cubic polynomial $f(x) = x^3 - x$, with 3 real roots $0, \pm 1$: Find an explicit interval containing each of the three fixed points of $N$ where the reasoning of Problem (1) applies. (See Figure 5.2 of notes or figure 2 of the next page).

   (d) Now choose the points $x = 0.4, 0.45, 0.455, 0.5.0.6$ where clearly $|N'(x)| > 1$, iterate $N$ starting at each of these points, and see what happens.

2. Rudin Chapter 7, exercise 4: For $x \in \mathbb{R}$, consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x}$$

   For which values of $x$ does it converge absolutely? On which intervals does it converge uniformly? On which intervals does it fail to converge uniformly? Is $f$ continuous wherever the series converges? Is $f$ bounded?

3. Rudin Chapter 7 exercise 7: For $n \in \mathbb{N}$ and $x \in \mathbb{R}$ put

$$f_n(x) = \frac{x}{1 + nx^2}.$$

   Show that $f$ converges uniformly to a function $f$, and that the equation $f'(x) = \lim f_n'(x)$ is correct for $x \neq 0$, but false for $x = 0$.

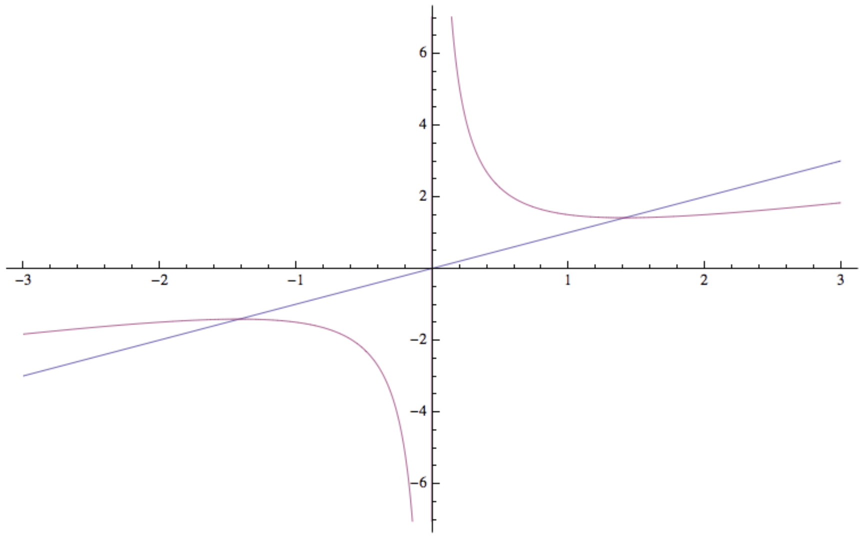


Figure 1: The Newton Map for $x^2 - 2$

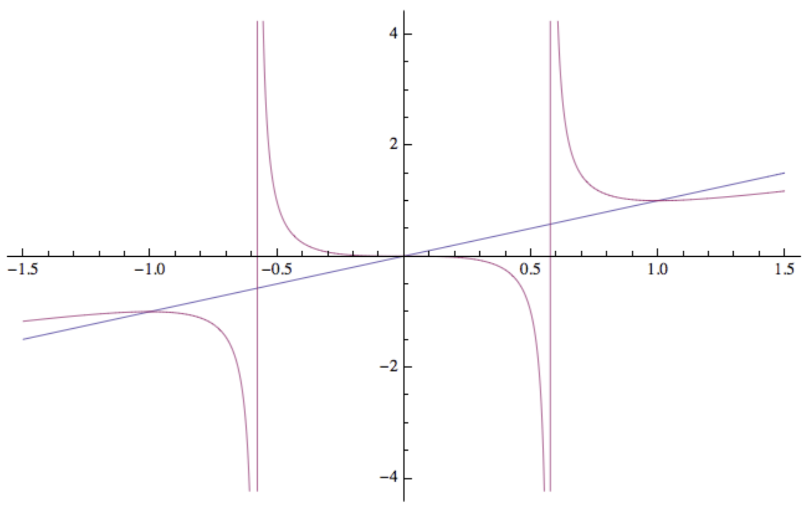


Figure 2: The Newton Map for $x^3 - x$