1. Let $a_1, b_1 > 0$ and define, for $n \geq 1$,

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n},$$

in other words, a_{n+1} is the *arithmetic mean* of a_n, b_n and b_{n+1} is the *geometric mean* of a_n, b_n. Therefore, by the inequality between arithmetic and geometric means, we have that $a_n \geq b_n$ for all $n \geq 2$.

   (a) Prove that the sequence $\{a_n\}$ is decreasing, the sequence $\{b_n\}$ is increasing, and for all $m < n$ we have

   $$b_m \leq b_n \leq a_n \leq a_m.$$

   Conclude that the numbers $A = \lim\{a_n\}$ and $B = \lim\{b_n\}$ exist. Then use either of the above recursion formulas to show that $B = A$. This number is called the *Arithmetic-Geometric Mean* of a and b.

   (b) Prove the inequality

   $$a_{n+1} - b_{n+1} \leq \frac{1}{2}\frac{(a_n - b_n)^2}{a_n + b_n}. \tag{1}$$

   (*Suggestion*: Prove first that if in (1) the denominator $(a_n + b_n)$ is replaced by $a_n + b_n + 2\sqrt{a_n b_n}$, then the two sides are actually equal. Then drop the third term in the denominator to get the inequality)

   (c) Deduce from (1) that the convergence is *quadratic* meaning that there is some constant C so that $a_{n+1} - b_{n+1} \leq C(a_n - b_n)^2$. This means, in particular, that once $a_n - b_n < \max\{1, 1/C\}$, (which must happen since $0 < a_n - b_n \to 0$) the convergence must be very fast. (*Suggestion*: Looking at (1), all you need is some constant $C_1 > 0$ so that $a_n + b_n \geq C_1$)

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3. (Do this but don't hand in) Observe that in both these problems you have quadratic convergence, which implies, roughly, that the number of correct digits should at least double after each iteration. Use a computer to calculate the Arithmetic-Geometric mean of $1,000$ and 1 by the iteration of problem 1, and to compute $\sqrt{2}$ by the iteration of problem 2, and check what happens to the number of correct decimal places. Do ten iterations to as much precision as your computer allows.

4. Recall the picture of the p- norms on $\mathbb{R}^2$ (over). Prove that the situation is as shown in the picture: $||x||_s < ||x||_r$ if $1 \leq r < s \leq \infty$, with equality if and only if one of the coordinates is zero, in other words, if $r < s$, then

$$(|x_1|^s + |x_2|^s)^{\frac{1}{s}} \leq (|x_1|^r + |x_2|^r)^{\frac{1}{r}}$$

with equality if and only if either $x_1 = 0$ of $x_2 = 0$. (*Suggestion*: since $||ax|| = |a|\ ||x||$, enough to prove this for $||x||_r = 1$.)

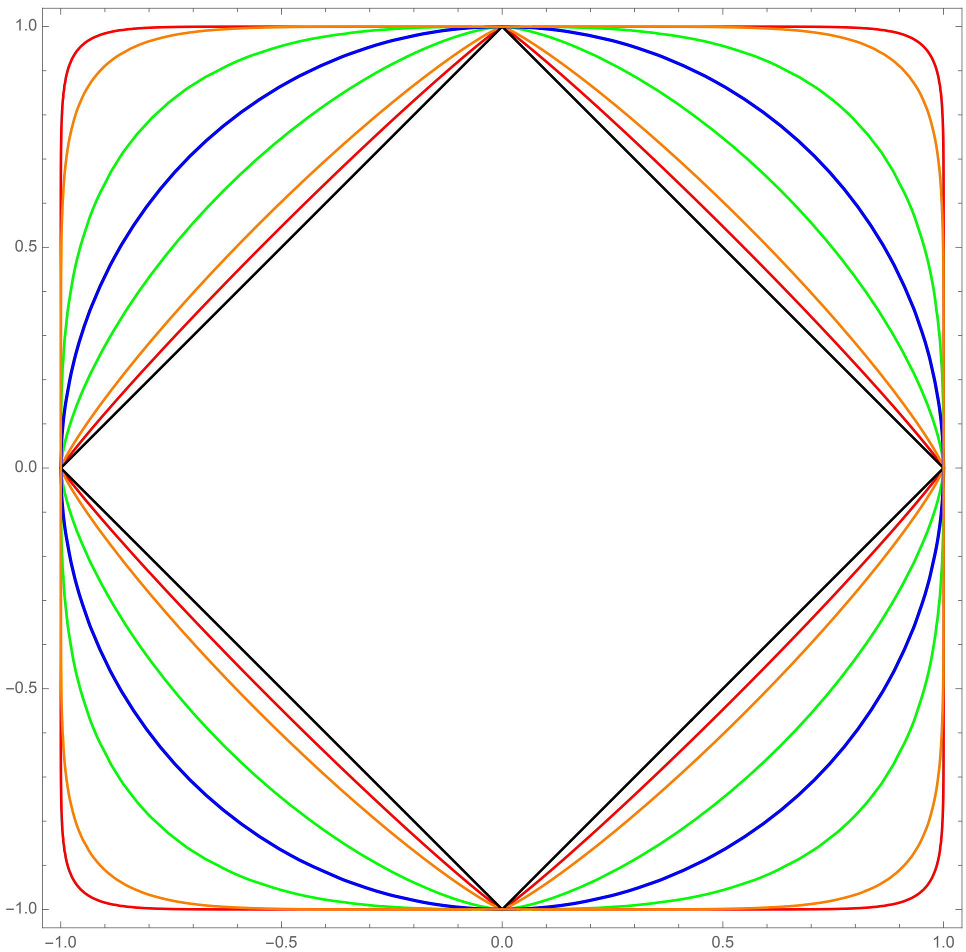

Figure 1: From inside: $p = 1, \frac{15}{14}, \frac{7}{6}, \frac{3}{2}, 2, 3, 7, 15, \infty$. Conjugate exponents with same color.