Math 5210 Homework 2, due January 28

1. Let $[a,b] \subset \mathbb{R}$ be an interval, and let $\mathcal{C}[a,b]$ be the set of continuous real valued functions on $[a,b]$. Then $\mathcal{C}[a,b]$ is a real vector space. Define an inner product on $\mathcal{C}[a,b]$ by

$$< f, g >= \int_a^b f(x)g(x)dx,$$

and define two norms

$$||f||_2 =< f, f >^{\frac{1}{2}}= \Big(\int_a^b f(x)^2 dx\Big)^{\frac{1}{2}},$$

and

$$||f||_1 = \int_a^b |f(x)|dx.$$

   (a) Prove the Schwarz inequality $< f, g >^2 \leq \; < f, f >< g, g >$ by the same method used in class to prove the Cauchy-Schwarz inequality in $\mathbb{R}^n$, namely find a formula for the difference

$$\int_a^b f(x)^2 dx \int_a^b g(x)^2 dx - \Big(\int_a^b f(x)g(x)dx\Big)^2$$

   as an integral ever the product $[a,b] \times [a,b]$. Then give and prove the necessary and sufficient condition for equality to hold in this inequality.

   (b) Prove that $||f||_1 \leq \sqrt{b-a}||f||_2$, and equality holds if and only if f is constant.

2. If V is a normed vector space, the distance between $x, y \in V$ is defined by $d(x,y) = ||y - x||$. The purpose of this exercise is to find conditions for equality in the triangle inequality in some of the normed spaces that we know. In other words, given $x \in V$, want to describe the following set:

$$E_x = \{y \in V : ||x|| = ||y|| + ||x - y||\} \quad = \quad \{y \in V : d(0,x) = d(0,y) + d(y,x)\}.$$

   Find this set for

   (a) $\mathbb{R}^2$ with $||x||_1 = |x_1| + |x_2|$. Fix $x = (x_1, x_2)$ in first quadrant. Draw a picture of E_x.

   (b) $\mathbb{R}^2$ with norm $||x||_\infty = \max\{|x_1|, |x_2|\}$. (*Suggestion*: see how the unit balls in the d_1 and d_∞ are related to get a hint as to how the equality sets for the triangle inequality are related)

   (c) $\mathcal{C}[a,b]$ with $||f||_2$

   (d) $\mathcal{C}[a,b]$ with $||f||_1$.

3. (Rudin, Chapter 3, problem 7): Prove that if $a_n \geq 0$ and $\sum_{i=0}^{\infty} a_n$ converges, then so does

$$\sum_{i=1}^{\infty} \frac{\sqrt{a_n}}{n}.$$