(1) Pugh, Exercise 8.

(2) Let $f_n : [0,1] \to \mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} 1 & \text{if } x = \frac{p}{q} \text{ fraction in lowest terms with } q \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Show that $f_n \uparrow f$, where

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise.} \end{cases}$$

Compute $\int f_n$, $\lim \int f_n$ and $\int f$ both for the Riemann integral and the Lebesgue integral. Use these answers to show that the following notions are *not* the same for the Riemann and Lebesgue integrals:

(a) Integrability.
(b) Monotone convergence theorem (Pugh, Theorem 15).
(c) Dominated convergence theorem (Pugh, Theorem 20).

(3) Let $f_n : [0,1] \to \mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} n^2 x & \text{if } 0 \leq x \leq \frac{1}{n} \\ 2n - n^2 x & \text{if } \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & \text{if } \frac{2}{n} \leq x \leq 1. \end{cases}$$

These are the "steeple functions" mentioned in the remark following Theorem 21 of Pugh (Fatou's lemma).

(a) Draw the graphs of the f_n and find $\lim f_n$.
(b) Compute $\int f_n$, $\lim \int f_n$ and $\int f$, and show that you get strict inequality in Fatou's lemma: $\int f < \liminf \int f_n$.
(c) Explain why this does not contradict the dominated convergence theorem.

(4) Rudin, Chapter 11, problem 5. This gives another example of strict inequality in Fatou's lemma.