Math 5210, Topics for Midterm 2

You should know the following definitions and theorems, and have some idea of how the theorems are proved and how they can be used. References are: N = notes, R = Rudin.

1. *The Contraction Mapping Theorem* (N §5, R 9.23): Let (X, d) be a complete metric space, and let $f : X \to X$ be a contraction. This means that there exists a constant $C < 1$ so that for all $x, y \in X$, $d(f(x), f(y)) \leq C\ d(x, y)$. Then f has a unique fixed point in X. This means that there is a unique point $x_0 \in X$ such that $f(x_0) = x_0$.

2. *Proof of the Contraction Mapping Theorem*: Recall that the proof is very simple: pick any point $x_1 \in X$ and make it the first entry of the sequence $\{x_n\}$ defined by $x_2 = f(x_1)$, $x_3 = f(x_2), \ldots, x_{n+1} = f(x_n), \ldots$. Then $d(x_{n+1}, x_{n+2}) < C\ d(x_n, x_{n+1}) < \ldots C^n d(x_2, x_1)$ which implies that $\sum_{n=1}^{\infty} d(x_{n+1}, x_n)$ converges, which implies that $\{x_n\}$ is a Cauchy sequence, hence converges. Its limit must necessarily be fixed by f.

3. Some applications of the contraction mapping theorem:

   (a) Examples of contractions: Suppose $f : [a, b] \to [a, b]$ and there is a constant $C < 1!$ such that $|f'(x)| < C$ for all $x \in [a, b]$. Then f is a contraction (with the same constant C). (N §3.4).

   (b) Newton's method: Suppose $f : \mathbb{R} \to \mathbb{R}$ is of class C^2. Define the *Newton map* associated to f by

   $$N(x) = x - \frac{f(x)}{f'(x)} \quad \text{defined on} \quad \{x : f'(x) \neq 0\}.$$

   The fixed points of N are the zeros of f: $N(x) = x$ if and only if $f(x) = 0$. If $f(x_0) = 0$, then, for moderately small δ, in the interval $|x - x_0| \leq \delta$ we get an upper bound for $|N'(x)| = |(f(x)\ f''(x))/(f'(x)^2)|$ by a constant $C < 1$, thus N is a contraction on $|x - x_0| \leq \delta$. (N §5.1).

   (c) If X is a compact metric space, the space $(C(X), d_\infty)$ of continuous $\mathbb{R}$-valued functions on X with $d_\infty(f, g) = ||f - g||_\infty = \max\{|f(x) = g(x)| : x \in X\}$ is a *complete* metric space. (N §3.2.3, R 7.15). Be familiar with this space, how to prove its completeness. In particular, be familiar with the fact that $f_n \to f$ in d_∞ if and only if $f_n \to f$ *uniformly* on X.

   (d) *Picard Iteration* The completeness of $(C(X), d_\infty)$ can be used to prove the existence and uniqueness theorem for solutions of the initial value problem to first order differential equation

   $$\begin{aligned} \frac{dx}{dt} &= f(t, x(t)) \\ x(0) &= x_0 \end{aligned} \tag{1}$$

   The theorem in question is:

Theorem 1 *Suppose that $U \subset \mathbb{R}^2$ is open and $f : U \to \mathbb{R}$ is continuous and satisfies a local, time - independent Lipschitz condition on U, meaning that on every closed sub-rectangle $R = [a, b] \times [c, d] \subset U$, $a, b, c, d \in \mathbb{R}$, there is a constant $c_R > 0$ so that $|f(t, x) - f(t, y)| \leq c_R\, |x - y|$ for all x, y, t so that (t, x) and (t, y) are in U.*
Then there exist numbers $a, b > 0$ so that the rectangle $[-a, a] \times [x_0 - b, x_0 + b] \subset U$ and so that (1) has a unique solution with graph contained in this rectangle, that is, $x(t)$ is defined for $|t| < a$ and satisfies $|x(t) - x_0| < b$ for $|t| < a$. Moreover, $x(t)$ is a continuously differentiable function of t.

Recall that this is proved by converting (1) to an integral equation

$$Px = x, \quad \text{where} \quad (Px)(t) = x_0 + \int_0^t f(\tau, x(\tau))d\tau. \tag{2}$$

and solving (2) by iteration as in the proof outlined above of the Contraction Mapping Theorem. See (N §5.2) for details. Make sure you can carry out an example of an iteration.

(e) *The Inverse Function Theorem* : First review the definition of differentiability for functions $f : U \to \mathbb{R}^n$ where $U \subset \mathbb{R}^m$ is an open set , the definition of the derivative df which is a linear transformation given by the Jacobian matrix, and the definition of the class C^1 of continuously differentiable functions. See (R 9.10 – 9.21, N §6). The statement of the Inverse Function Theorem is:

Theorem 2 *Let $U \subset \mathbb{R}^n$ be an open set, let $f : U \to \mathbb{R}^n$ be of class C^1 (continuously differentiable). Let $x_0 \in U$ and suppose that $d_{x_0}f : \mathbb{R}^n \to \mathbb{R}^n$ is invertible. Then there exist neighborhoods $N(x_0) \subset U$ of x_0 and $N(y_0)$ of $y_0 = f(x_0)$ so that $f(N(x_0)) = N(y_0)$ and the restriction of f to $N(x_0)$, denoted $f|_{N(x_0)} : N(x_0) \to N(y_0)$ is bijective, so it has an inverse. This inverse map $(f|_{N(x_0)})^{-1} : N(y_0) \to N(x_0)$ is also of class C^1.*

For proofs, see (N §6, R 9.24).

(f) *The Implicit Function Theorem*: See (R 9.26 to 9.29). To avoid some of the complicated notation in (R 9.26 - 9.29), let's consider the special case of a real valued function on an open set $U \subset \mathbb{R}^{n+1}$. To have a reasonable notation to state the theorem, write $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ and points in $\mathbb{R}^{n+1}$ as (p, r), where $p \in \mathbb{R}^n$ and $r \in \mathbb{R}$. This means, write $(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n) \times (x_{n+1})$, use shorthand $p = (x_1, \ldots, x_n)$ and $r = (x_{n+1})$. Using this notation, the theorem says:

Theorem 3 *Let $g : U \to \mathbb{R}$ be of class C^1, and suppose that $(p_0, r_0) \in U$ and that $\frac{\partial g}{\partial x_{n+1}}(p_0, r_0) \neq 0$. Then there are $a, b > 0$ so that $N = B(p_0, a) \times B(r_0, b) \subset U$, and there is a function $\phi : B(p_0, a) \to B(r_0, b)$ of class C^1 so that*

$$Z = \{(p, r) \in N : g(p, r) = 0\} = \{(p, \phi(p)) : p \in B(p_0, a)\}. \tag{3}$$

In other words, we can locally solve for x_{n+1} as a function of $x_1, \ldots, x_n$ (this is the "implicit function"), or, in the shorthand notation, $r = \phi(p)$. In more precise terms, if we let Z be the zero set of g, then (3) says that $Z \cap N$ is the graph of ϕ.

Proof: Define $F : U \to \mathbb{R}^{n+1}$ by $F(p, r) = (p, g(p, r))$. Then the derivative $d_{(p,r)}F$ is given by the Jacobian matrix, which is the $(n + 1)$ by $(n + 1)$ matrix

$$\begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \\ g_1 & g_2 & \dots & g_n & g_{n+1} \end{pmatrix},$$

where g_i stands for $\frac{\partial g}{\partial x_i}$.

Since the determinant of this matrix is clearly g_{n+1}, which, by assumption, doesn't vanish at (p_0, r_0), this matrix is invertible at (p_0, r_0). By the Inverse Function Theorem (2), F is invertible in a neighborhood N of (p_0, r_0), which we can choose to be a product of balls as in the N in the statement of the theorem. Let $N' = F(N)$ be the neighborhood of $F(p_0, r_0) = (p_0, 0)$ o which the local inverse $\Phi : N' \to N$ of F is defined.

Let's also write (q, s), where $q \in \mathbb{R}^n$ and $s \in \mathbb{R}$ for points in the target. Since $F(p, r) = (p, g(p, r))$, it follows that $\Phi(q, s) = (q, \psi(q, s))$ for some $\psi : N' \to \mathbb{R}$.
Since $(p, r) \in Z$ if and only if $g(p, r) = 0$, we see that $(p, r) \in Z$ if and only if $F(p, r) = (p, 0)$, therefore if and only if $(p, r) = \Phi(p, 0) = (p, \psi(p, 0))$. So, if we define $\phi(p) = \psi(p, 0)$, then Z is the graph of ϕ, as desired, and the proof is complete;
Remark: Geometrically, F "straightens " Z into the "plane" $\mathbb{R}^n \times 0$.

4. *Other Topics*: These are closely related to the topics discussed above and are required for the details of the proofs:

   (a) Uniform convergence of sequences of functions: (R 7.1 to 7.16).

   (b) Linear transformations, their norms, continuity of inversion: (R 9.1 to 9.8).