Math 5210, Topics for the Final Exam

The final exam will be comprehensive, but with more emphasis on the more recent topics. I will ask questions from the material already covered in the first two midterms, as well as in the new material, roughly half old and half new. References are: N = notes, P = Pugh, chapter 6, R = Rudin.

1. Basic material on metric spaces: Definitions, examples, Cauchy sequences, completeness.

2. Inequalities: Cauchy Schwarz and related ones, Jensen's inequality (convexity) (N4).

3. Normed spaces: $\mathbb{R}^n$ and $\mathbb{R}^\infty$, $C[a,b]$ with the p- norms, $1 \leq p \leq \infty$. (N4)

4. Completeness of $C[a,b], ||f||_p$ for $p = \infty$, incompleteness for $1 \leq p < \infty$ (N3).

5. Contraction mapping theorem and applications. (R9, N5)

6. Basic material on differentiable maps from $\mathbb{R}^m$ to $\mathbb{R}^n$ (R9).

7. Inverse and implicit function theorems (R9, N6).

8. Spaces of continuous functions:

   (a) Equicontinuous families, compact subsets of $C(X)$ for X compact metric (R7.22 to 7.25).

   (b) Arzela - Ascoli theorem: $\{f_n\}$ a bounded, equicontinuous sequence in $C(X)$, has a uniformly convergent subsequence. (R7.25).

   (c) Weierstrass approximation theorem: Polynomials are dense in $C[0,1]$. (R7.26)

9. Lebesgue Theory: Definitions

   (a) Lebesgue outer measure m^* in $\mathbb{R}$ and $\mathbb{R}^2$, sets of measure zero. (P1):

   $$m^*(E) = \inf\{\sum_i |I_i| : \{I_i\} \text{ countable collection of open intervals with } \cup I_i \supset E\}$$

   (b) Abstract outer measure $\omega : 2^M \to [0.\infty]$, where M is any set: a monotone, countably subadditive function with $\omega(\emptyset) = 0$ (P2).

   (c) Sigma algebra: a collection of subsets of M containing $\emptyset$ and closed under complement and countable union. Measure on a sigma-algebra: a monotone, countably *additive* $[0,\infty]$-valued function on the sigma algebra, assigning 0 to $\emptyset$. (P2)

   (d) Given a set M and an abstract outer measure on 2^M, definition of measurable sets and of the measure of a measurable set. (P2) A set $E \subset M$ is measurable if and only if, for all $X \subset M$,

   $$\omega(X) = \omega(X \cap E) + \omega(X \cap E^c).$$

   If E is measurable, its measure is its outer measure.

(e) In $\mathbb{R}$ or $\mathbb{R}^2$: a G_δ-set is a countable intersection of open sets, an F_σ-set is a countable union of closed sets (P3).

(f) (P4) For any function $f : \mathbb{R} \to [0, \infty)$, define its *undergraph*

$$\mathcal{U}(f) = \{(x, y) \in \mathbb{R}^2 : 0 \leq y < f(x)\}$$

Then

i. f is *Lebesgue measurable* if and only if $\mathcal{U}(f)$ is a Lebesgue measurable subset of $\mathbb{R}^2$.

ii. If f is Lebesgue measurable, its Lebesgue integral is defined to be

$$\int f = m_2(\mathcal{U}(f))$$

where m_2 denotes Lebesgue measure in $\mathbb{R}^2$ (two-dimensional Lebesgue measure).

(g) (P5) Upper and lower Lebesgue sums for a partition Y of $[0, \infty)$, $0 = y_0 < y_1 < y_2 < \ldots \to \infty$:

$$\underline{L}(f, Y) = \sum_{i=1}^{\infty} y_{i-1} m(X_i), \quad \overline{L}(f, Y) = \sum_{i=1}^{\infty} y_i m(X_i),$$

where $X_i = f^{-1}([y_{i-1}, y_i))$.

10. Lebesgue theory: Theorems.

(a) Countable subsets of $\mathbb{R}$ or $\mathbb{R}^2$ have measure zero (P1).

(b) Lebesgue outer measure of an interval is its length, of a rectangle is its area. (P1)

(c) Limit properties of measures: upward and downward measure continuity theorems (P2): If E_n are measurable, and $E_n \uparrow E$, then E is measurable and $\omega(E_n) \uparrow \omega(E)$. If $E_n \downarrow E$ and $\omega(E_1) < \infty$, then E is measurable and $\omega(E_n) \downarrow \omega(E)$.

(d) In $\mathbb{R}$ intervals are measurable, sets of measure zero are measurable (P3).

(e) Regularity of Lebesgue measure: if E is measurable set in $\mathbb{R}$ or $\mathbb{R}^2$, there exist a G_δ set G, an F_σ-set F so that $F \subset E \subset G$ and $m(G \setminus F) = 0$ (P3).

(f) Convergence Theorems: Monotone Convergence, Dominated Convergence (P4). Know how these can fail for Riemann integrals (homework).

(g) Fatou's Lemma (P4). Know examples that show inequality can be strict (homework).

(h) If f is Lebesgue integrable, then $\underline{L}(f, Y) \uparrow \int f$, as mesh of $Y \to 0$ (P5).