Let (X, d) be a metric space. We will use the following definitions (see Rudin, chap 2, particularly 2.18)

1. Let $p \in X$ and $r \in \mathbb{R}$, $r > 0$, The *ball of radius r centered at p* is
$$B(p, r) = \{q \in X : d(p, q) < r\}.$$
This is the same as the *neighborhood* $N_r(p)$ in Rudin.

2. If $E \subset X$, then
   (a) A point $p \in E$ is called an *interior point* of E if there is an $r > 0$ so that $B(p, r) \subset E$.
   (b) E is *open* if every point of E is an interior point of E. Explicitly, this means that for every $p \in E$ there is an $r > 0$ so that $B(p, r) \subset E$.
   (c) E is *closed* if its complement $X \setminus E$ is open.
   (d) A point $p \in X$ is a *limit point* of E if for every $r > 0$, there is a point $q \neq p$ such that $q \in B(p, r) \cap E$. *Note*: p may or may not be an element of E.
   (e) If $p \in E$ and p is not a limit point of E, then p is called an *isolated point* of E. this means: p is an isolated point of E if and only if $p \in E$ and there exists an $r > 0$ so that $B(p, r) \cap E = \{p\}$.
   (f) E is *perfect* if E is closed and every $p \in E$ is a limit point of E.
   (g) E is *bounded* if there is a real number $M > 0$ and $p \in X$ so that $E \subset B(p, M)$. In other words, there is an $M > 0$ and $p \in X$ so that $d(p, q) < M$ for all $q \in E$.

3. Some easy but important theorems:
   (a) For all $p \in X$ and all $r > 0$, $B(p, r)$ is open in X. Also $\{q \in X : d(p, q) > r\}$ is open in X.

   Proof: Use the triangle inequality.
   (b) *$E \subset X$ is closed if and only if it contains all its limit points.*

   Proof: Observe that $p \in X \setminus E$ is a limit point of E if and only if for all $r > 0$, $B(p, r)$ contains points of E, in other words, $B(p, r)$ is not contained in $X \setminus E$, in other words, if and only if p is not an interior point of $X \setminus E$, that is, if and only if $X \setminus E$ is not open, that is, by definition, if and only if E is not closed.
   (c) *The collection of open sets is closed under the operations of arbitrary union and finite intersection.* Explicitly (see Rudin 2.24 for a proof):
      i. Let $\{U_\alpha\}_{\alpha \in A}$ be a collection of open sets in X. Then $\cup_{\alpha \in A} U_\alpha$ is open.
      ii. Let $\{U_i\}_{i=1}^n$ be a finite collection of open sets in X. Then $\cap_{i=1}^n U_i$ is open.

   And by taking complements get the corresponding statements for closed sets (Rudin 2.24):
      i. Let $\{F_\alpha\}_{\alpha \in A}$ be a collection of closed sets in X. Then $\cap_{\alpha \in A} F_\alpha$ is closed.

ii. Let $\{F_i\}_{i=1}^n$ be a finite collection of closed sets in X. Then $\cup_{i=1}^n F_i$ is closed.

(d) These theorems allow us to give the following definitions: Given $E \subset X$, let

$$E^o = \cup\{U \subset E : U \text{ is open in } X\}$$

and let

$$\overline{E} = \cap\{F : F \text{ is closed in } X \text{ and } E \subset F\}.$$

Then E^o is an open set, it is called the *interior of* E, and it is characterized by the property that it is the *largest open set contained in* E. This means that if U is open and $U \subset E$, then $U \subset E^0$. An alternative characterization (often taken as the definition) is that E^o is the *set of all interior points of* E, in other words,

$$E^0 = \{p \in E : \text{ there exists } r > 0 \text{ such that } B(p,r) \subset E\}.$$

Note that E^o may be empty.

Similarly, the set $\overline{E}$ is closed, it is called the *closure of* E, and it is characterized by the property that it is the *smallest closed set containing* E. This means that if F is closed and $E \subset F$, then $\overline{E} \subset F$. An alternative characterization (often taken as the definition) is that $\overline{E}$ is the union of E with the collection of its limit points. To see the equivalence, since we have seen that $p \in X \setminus E$ is a limit point of E if and only if it is not interior to $X \setminus E$, we wee that the set E union its limit points can be written as

$$E \cup \{p \in X \setminus E : p \text{ is a limit point of } E\} = E \cup ((X \setminus E) \setminus (X \setminus E)^0) = X \setminus (X \setminus E)^o.$$

Now since $(X \setminus E)^o$ is the largest open set contained in $X \setminus E$, it follows that $X \setminus (X \setminus E)^o$ is the smallest closed set containing E, hence

$$\overline{E} = X \setminus (X \setminus E)^o$$

and the two characterizations of the closure coincide.

Observe that we may have $\overline{E} = X$. In this case we say that *E is dense in X*.

(e) *Continuous Functions*: Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a function.

i. f is *continuous* if and only if, for every $x \in X$ and every $\epsilon > 0$ there exists $\delta > 0$ so that for all $y \in X$ with $d_X(x, y) < \delta$, we have $d_Y(f(x), f(y)) < \epsilon$.

ii. f is *uniformly continuous* if and only if, for every $\epsilon > 0$ there is a $\delta > 0$ so that for all $x, y \in X$ with $d_X(x, y) < \delta$ we have $d_Y(f(x), f(y)) < \epsilon$.

iii. f is a *Lipschitz function* if and only if there exists a constant $C > 0$ (called *a Lipschitz constant* so that for all $x, y \in X$ we have $d_Y(f(x), f(y)) \leq C d_X(x, y)$. The infimum of all Lipschitz constants is called *the* Lipschitz constant of f. (Notes 3.4). Differentiable mappings with bounded derivative provide a large class of examples of Lipschitz maps (Notes 3.4, example 3.1).

Clearly Lipschitz implies uniformly continuous (given ϵ, choose $\delta = \epsilon/C$) and uniformly continuous implies continuous (with $\delta = \delta(\epsilon)$, independent of x). These implications cannot be reversed:

i. Let $f : [0, \infty) \to [0, \infty)$ be defined by $f(x) = x^2$. Then $f(x) - f(y) = (x - y)(x + y)$. For every $\delta > 0$, if $x > 1/\delta$ and $y = x + \delta/2$, then $|f(x) - f(y)| = (x + y)|x - y| > (2/\delta)|x - y| = (2/\delta)(\delta/2) = 1$, so for $\epsilon = 1$ we cannot find any δ that works for all x, y. Thus f is not uniformly continuous.

ii. Let $f : [0, 1] \to [0, 1]$ be defined by $f(x) = \sqrt{x}$. Then f is uniformly continuous. This follows from general principles, since it is a continuous function on a compact metric space, see below (and Rudin 4.19). More directly, if $x > y$, it is easy to see that $0 < \sqrt{x} - \sqrt{y} < \sqrt{x - y}$ (square both sides and manipulate a bit), so $|f(x) - f(y)| \leq \sqrt{|x - y|}$, so given $\epsilon > 0$ if we let $\delta = \epsilon^2$, then $|x - y| < \delta$ implies $|f(x) - f)y)| < \sqrt{\delta} = \epsilon$, so f is uniformly continuous.
But f is not Lipschitz: $|f(x) - f(0)| = |x|/\sqrt{x}$. Given any $C > 0$, if $\sqrt{x} < 1/C$, then $|f(x) - f(0)|/|x - 0| > C$, so f is not Lipschitz.

The definition (i) of continuous is equivalent to two other very convenient definitions:

i. f is continuous if and only if for every open set $U \subset Y$, $f^{-1}(U)$ is open in X.

ii. f is continuous if and only if for every closed set $F \subset Y$, $f^{-1}(F)$ is closed in X.

Clearly the last two definitions are equivalent. (Rudin 4.8). The $\epsilon - \delta$ definition of continuity says that for every $x \in X$, and $\epsilon > 0$ there is a $\delta > 0$ so that $f(B(x, \delta)) \subset B(f(x), \epsilon)$. If we let $U \subset Y$ be open and take $x \in f^{-1}(U)$, then $f(x) \in U$ and there is $\epsilon > 0$ so that $B(f(x), \epsilon) \subset U$. Then $B(x, \delta) \subset f^{-1}(U)$, so $f^{-1}(U)$ is open. The opposite direction is similar.

One advantage of the alternative definitions of continuity is that it makes theorems as the following obvious:

Theorem: If $g : X \to Y$ and $f : Y \to Z$ are continuous, then the composition $f \circ g : X \to Z$ is continuous.

Proof: If $U \subset Z$ is open, then $f^{-1}(U) \subset Y$ is open, then $g^{-1}(f^{-1}(U) \subset X$ is open. But $g^{-1}(f^{-1}(U)) = (f \circ g)^{-1}(U)$, so $f \circ g$ is continuous.

Note: There are no analogous alternative definitions for uniform continuity (nor for Lipschitz).

4. *Compactness*

(a) Let $E \subset X$. A collection $\{U_\alpha\}_{\alpha \in A}$ of open subsets of X is called an *open cover of* E if and only if $E \subset \cup_{\alpha \in A} U_\alpha$.

(b) Given an open cover as above, a *subcover* means a subcollection $\{U_\beta\}_{\beta \in B}$ where $B \subset A$ so that the subcollection still covers E: $E \subset \cup_{\beta \in B} U_\beta$. A subcover is called a *finite subcover* if B is a finite set.

(c) $K \subset X$ is called *compact* if and only if every open cover has a finite subcover.

(d) Compact sets are bounded;

Proof: Let $K \subset X$ be compact, fix $p \in X$ and consider the open cover $\{B(p, n) : n \in \mathbb{N}\}$ of K.

(e) If $K \subset X$ is compact, then K is closed (see Rudin 2.34);

(f) Closed subsets of compact sets are compact (Rudin 2.35)

(g) In $\mathbb{R}^n$, $K \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded. (Rudin 2.41).

(h) In many metric spaces there are closed, bounded sets that are not compact. See, for example, (3.3) of the Notes, for examples in infinite dimensional function spaces.

(i) $f : X \to Y$ continuous and $K \subset X$ compact, then $f(K)$ is compact. (Rudin 4.14)

(j) X compact, $f : X \to \mathbb{R}$ continuous, then f has a maximum and minimum in X: There exist points $p_0, p_1 \in X$ so that $f(p_0) \leq f(p) \leq f(p_1)$ for all $p \in X$. (Rudin 4.16).

(k) X compact, $f : X \to Y$ continuous, then f is uniformly continuous. (Rudin 4.19).

5. *Subspaces of a Metric Space*

If (D, d) is a metric space and $E \subset X$, then E is a metric space with distance $d_E =$ restriction of d_X. So we can define open sets, closed sets, etc for the metric space E.

Theorem $A \subset E$ is open if and only if there exists an open set $U \subset X$ so that $A = E \cap U$. Similarly, A is closed in E if and only if there exists a closed set $F \subset X$ such that $A = E \cap F$

Proof Follows from the observation that for all $p \in E$, $B_E(p, r) = B_X(p, r) \cap E$, where B_E, B_X mean the open balls in $(E, d_E), (X, d_X)$ respectively. Suppose $A \subset E$ is open: for all $p \in E$ there exists $r(p) > 0$ such that $B_E(p, r(p)) \subset E$. Let $U = \cup_{p \in E} B_X(p, r(p))$. Then U is open in X and $U \cap E = A$. Conversely, if $U \subset X$ is open and $U \cap E = A$, then given any $p \in A$ there is an $r > 0$ so that $B_X(p, r) \subset U$. Since $U \cap E = A$, we have $B_E(p, r) = B_X(p, r) \cap E \subset U \cap E = A$, thus A is open in E.

6. *Connected sets*:

Definition: A metric space X is *disconnected* $X = U \cup V$ where U, V are open, $U \cap V = \emptyset$, and $U \neq \emptyset, V \neq \emptyset$. A metric space is *connected* if and only if it is not disconnected.

This definition applies to any subset $E \subset X$ by using open sets in E. This definition is equivalent to Rudin's 2.45: $A, B \subset E$ are *separated* means $E = A \cup B$ so that $A \cap \overline{B} = \emptyset$ and $\overline{A} \cap B = \emptyset$, both $A, B \neq \emptyset$. Then A, B are open in E because $A = (X \setminus \overline{B}) \cap E$, thus is open in E, same for B, and A, B are disjoint. Converse is similar.

Observe that could replace "open" by "closed" for the two sets in the definition of disconnected.

Another equivalent defnition: X is connected if and only if the only subsets of X that are both open and closed are X and $\emptyset$.

(a) The connected subsets of $\mathbb{R}$ are precisely the intervals. (Rudin 2.47).

(b) $f : X \to Y$ continuous, $E \subset X$ connected, then $f(E)$ is connected. (Rudin 4.22)

(c) $f : [a, b] \to \mathbb{R}$ continuous, then for every d between $f(a)$ and $f(b)$ there exists $c \in [a, b]$ with $f(c) = d$. (Rudin 4.23)

7. *Complete Metric Spaces* Rudin 3.12. $\mathbb{R}$ is compete.

A compact metric space is complete (Rudin 3.11)

8. *Normed Vector Spaces* (Notes Chapter 4).

   (a) $\mathbb{R}^n$ with the norms $||x||_1, ||x||_2, ||x||_\infty$ is a normed vector space.

   (b) These norms on $\mathbb{R}^n$ are equivalent, $\mathbb{R}^n$ is *complete* in any of these norms.

   (c) These norms are also defined in the space $\mathbb{R}^\infty = \{(x_1, x_2, \ldots, 0, 0, \ldots) : x_i \in \mathbb{R}^n$, only finitely many $x_i \neq 0\}$, but they are no longer equivalent. $\mathbb{R}^\infty$ is *not* complete in any of these norms.

9. *Inequalities*(Notes Chapter 4)

   (a) Convex functions: Continuous function $\phi : \mathbb{R} \to \mathbb{R}$ (or $I \to \mathbb{R}$ where I is an interval) so that for all x, y, $\phi(\frac{x+y}{2}) \leq \frac{\phi(x)+\phi(y)}{2}$.

   (b) Cauchy-Schwarz inequality, case of equality.

   (c) Jensen's inequality.

   (d) Inequality between arithmetic and geometric means: if $x_1, \ldots x_n > 0$, then $(x_1 \ldots x_n)^{1/n} \leq (x_1 + \ldots x_n)/n$, with equality if and only if all the x_i are equal.